Modification of classical Spitzer ion–electron energy transfer rate for large ratios of ion to electron temperatures

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Corrections to the classical Spitzer heat transfer rate between ions and electrons are calculated for the case when the ion temperature T_i is significantly higher than the electron temperature T_e. It is found that slow electrons are partially depleted by their interactions with the ions, resulting in a decrease in the heat transfer in comparison with the Spitzer rate, which assumes perfectly Maxwellian electrons. The heat transfer steadily decreases from the classical value as T_i/T_e increases; for T_i/T_e values of several hundred, the heat transfer rate drops to around 60%–80% of the Spitzer result. A useful expression for the heat transfer correction factor in the case when all of the ion species are at the temperature T_i is found to be $P_{ie}/(P_{ie})_{\text{Spitzer}} \approx [1+(m_e/m_i)(T_i/T_e)]^{3/2} \exp\{-[3.5\Sigma_i(Z_i^2 n_i/n_e)(m_e/m_i)(T_i/T_e)]^{2/3}\}$. This expression is quite accurate for values of $\Sigma_i(Z_i^2 n_i/n_e)(m_p/m_i)(T_i/T_e)$ less than about 50 (where m_p is the proton mass), although it underestimates the heat transfer rate for larger values of T_i/T_e, and one must resort to the more accurate but more complex analytical results derived in the paper. In the event that the ion distribution is non-Maxwellian, T_i in the correction factor should be replaced by $2\langle E_i\rangle/3$, where $\langle E_i\rangle$ is the mean ion energy. © *1995 American Institute of Physics.*

I. INTRODUCTION

Rosenbluth[1,2] has shown that natural interactions of electrons with ions tend to cause a passive depletion of some of the slow electrons which promote ion–electron heat transfer, thereby decreasing the ion–electron heat transfer rate from its classical Spitzer value.[3,4] Unfortunately, Rosenbluth's derivation assumed that the ions were Maxwellian, the electrons were nearly Maxwellian, and the ion thermal velocity was much less than the electron thermal velocity. The object of this study is to broaden the scope of the derivation to cover even highly non-Maxwellian distribution functions and temperature regimes in which the mean ion velocity starts to approach the average electron velocity.

In addition to being a useful addition to the fundamental plasma physics of ion–electron heat transfer, this study should have several practical applications. One of the most important applications lies in evaluating the performance of advanced-fuel reactor concepts such as inertial-electrostatic confinement fusion[5] and migma,[6] which are intended to operate with non-Maxwellian ions that are at much higher energies than the mean electron energy. If the ion–electron heat transfer could be reduced appreciably from the classical Spitzer value, the electron temperature, and thus bremsstrahlung and synchrotron radiation losses, would be substantially reduced, and as a result the performance of such reactor systems would be significantly improved.

Before the main results of the paper are presented, Sec. II will offer a brief and fairly intuitive look at the ion–electron heat transfer problem. Then Sec. III will present much more detailed and rigorous calculations which should give a good description of the problem under a wide array of possible conditions (e.g., various types of ion velocity distributions, temperature ranges, etc.). Finally, Secs. IV and V will apply these general results to the specific cases in which the ion distributions are Maxwellian and monoenergetic, respectively, and derive simple approximate answers as well as more accurate analytical results.

II. BACKGROUND

Before presenting a detailed analysis of the ion–electron heat transfer problem, it is worthwhile to consider the more qualitative results offered by a much simpler model.

The energy exchange time between a test particle of velocity v and background particles with a Maxwellian velocity distribution characterized by the thermal velocity $v_t' \equiv \sqrt{2T'/m'}$ is defined[4] as

$$t_E \equiv \frac{m^2 v^3}{16\pi Z^2 Z'^2 e^4 n' \ln \Lambda} \times \frac{(v/v_t')^2}{[\operatorname{erf}(v/v_t') - (v/v_t')\operatorname{erf}'(v/v_t')]}, \tag{1}$$

in which the error function is

$$\operatorname{erf}(w) \equiv \frac{2}{\sqrt{\pi}} \int_0^w e^{-w'^2}\, dw', \tag{2}$$

the definition $\operatorname{erf}'(x) \equiv d[\operatorname{erf}(x)]/dx$ has been made, and in all other cases the primes denote the background particles as opposed to the test particle.

One finds the following electron–ion collision time t_E^{ei} and electron–electron collision time t_E^{ee} for electrons of velocity v in the limit $v_{ti} \ll v \ll v_{te}$:

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$$t_E^{ei} \equiv \frac{m_e^2 v^3}{16\pi Z_i^2 e^4 n_i \ln\Lambda}\,\frac{v^2}{v_{ti}^2}; \tag{3}$$

$$t_E^{ee} \equiv \frac{m_e^2 v^3}{16\pi e^4 n_e \ln\Lambda}\,\frac{3\sqrt{\pi}\,v_{te}}{4v}. \tag{4}$$

In the case of electrons for which $t_E^{ei} < t_E^{ee}$, collisions with the ions will tend to have a greater effect than collisions with the faster electrons. As may be seen from the energy exchange times, this constraint is satisfied for electrons whose speeds are less than a certain critical velocity v_c,

$$v^3 < \frac{3\sqrt{\pi}}{4}\,\frac{Z_i^2 n_i}{n_e}\,v_{ti}^2 v_{te} \equiv v_c^3. \tag{5}$$

Now the form of the modification to the Spitzer heat transfer rate may be obtained in a straightforward and intuitive manner. The power transferred from the ions to the electrons is essentially proportional to the number of electrons moving more slowly than the ions. Since it is assumed that $v_{ti} \ll v_{te}$, the energy transfer rate P_{ie} will be proportional to $f_e(0)$, the value of the electron velocity distribution at $v=0$.

For $v > v_c$ electron–electron collisions dominate and the electron distribution assumes what is essentially its usual Maxwellian distribution,

$$f_e(v) \propto \exp\left(-\frac{v^2}{v_{te}^2}\right) \quad (\text{for } v > v_c). \tag{6}$$

On the other hand, below the critical velocity the dominance of collisions with ions tends to upscatter some of the electrons to higher energies and thereby flatten out the electron distribution at a constant value, namely its value at the critical velocity:

$$f_e(v) \propto \exp\left(-\frac{v_c^2}{v_{te}^2}\right) \quad (\text{for } v < v_c). \tag{7}$$

Since $P_{ie} \propto f_e(0)$, one may see that the actual heat transfer rate in comparison with its classical Spitzer value is

$$\frac{P_{ie}}{(P_{ie})_{\rm Spitzer}} = \exp\left(-\frac{v_c^2}{v_{te}^2}\right) = \exp\left\{-\left(C\,\frac{Z_i^2 n_i}{n_e}\,\frac{m_e}{m_i}\,\frac{T_i}{T_e}\right)^{2/3}\right\}. \tag{8}$$

While the above calculation yields a value of $C = 3\sqrt{\pi}/4$, the true value of C cannot be found from this simple calculation. This limitation is caused by the uncertainty in the precise velocity at which the electron distribution may be considered to flatten out. All that can be said for now is that C appears to be a constant of order unity.

Having taken this first enlightening look at the problem, one may now appeal to more detailed calculations to ascertain the accuracy of this initial computation, determine the actual value of C, and extend the analysis to other cases not covered in this simple example.

It should be remarked from the outset that only collisional interspecies energy transfer will be considered. Various instabilities which might be driven by substantial deviations of the plasma from thermodynamic equilibrium and which would further promote energy transfer will be ignored; thus these calculations will serve to set a lower bound on the ion–electron heat transfer rate.

III. GENERAL DESCRIPTION OF THE PROBLEM

In this section will be presented a description of how the Fokker–Planck collision operator may be applied to the present problem in order to obtain the equilibrium particle distribution functions and the interspecies energy exchange rate.

A. Rosenbluth potentials for general isotropic distributions

Consider the distribution function f_α for a given particle species α; the distribution function is normalized such that

$$\int d^3\mathbf{v}\, f_\alpha(\mathbf{v}) = n_\alpha. \tag{9}$$

As presented by Rosenbluth,[7,8] the collisionally induced evolution of the particle distribution functions is governed by the Fokker–Planck collision operator,

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm col} = -\sum_\beta \Gamma_{\alpha\beta}\nabla_{\mathbf{v}}\cdot\left[f_\alpha \nabla_{\mathbf{v}} h_{\alpha\beta} - \frac{1}{2}\,\nabla_{\mathbf{v}}\cdot(f_\alpha \nabla_{\mathbf{v}}\nabla_{\mathbf{v}} g_{\alpha\beta})\right] \equiv \sum_\beta C_{\alpha\beta} \equiv -\nabla_{\mathbf{v}}\cdot\sum_\beta \mathbf{J}_{\alpha\beta}, \tag{10}$$

in which $C_{\alpha\beta}$ is the collision operator just between two species α and β, $\mathbf{J}_{\alpha\beta}$ is the collisional velocity–space flux, the sums over all β include $\beta=\alpha$,

$$\Gamma_{\alpha\beta} \equiv \frac{4\pi Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda}{m_\alpha^2}, \tag{11}$$

and the Rosenbluth potentials $h_{\alpha\beta}$ and $g_{\alpha\beta}$ are defined as

$$h_{\alpha\beta}(\mathbf{v}) \equiv \frac{m_\alpha + m_\beta}{m_\beta}\int d^3\mathbf{u}\,\frac{f_\beta(\mathbf{u})}{|\mathbf{v}-\mathbf{u}|}; \tag{12}$$

$$g_{\alpha\beta}(\mathbf{v}) \equiv \int d^3\mathbf{u} f_\beta(\mathbf{u})|\mathbf{v}-\mathbf{u}|, \tag{13}$$

with the useful relation

$$h_{\alpha\beta} = \left(\frac{m_\alpha + m_\beta}{2m_\beta}\right)\nabla_v^2 g_{\alpha\beta}. \tag{14}$$

For isotropic velocity distributions, the Rosenbluth potentials may be rewritten, as shown in Eqs. (A1) and (A2) of Appendix A, producing the results:

$$h_{\alpha\beta}(v) = 4\pi\left(\frac{m_\alpha + m_\beta}{m_\beta}\right)\left[\int_0^v du\, f_\beta(u)\left(\frac{u^2}{v} - u\right) + \int_0^\infty du\, f_\beta(u)u\right]; \tag{15}$$

$$g_{\alpha\beta}(v)=\frac{4\pi}{3}\left[\int_0^v du\, f_\beta(u)\left(\frac{u^4}{v}+3u^2v-3u^3-uv^2\right)\right.$$
$$\left.+\int_0^\infty du\, f_\beta(u)(uv^2+3u^3)\right]. \tag{16}$$

B. Interspecies heat transfer rate

Define the rate of energy transfer per volume from the α species to the β species to be $P_{\alpha\beta}$:

$$P_{\alpha\beta}=-\int d^3\mathbf{v}\left(\frac{1}{2}m_\alpha v^2\right)C_{\alpha\beta}. \tag{17}$$

By using the definition of the Fokker–Planck collision operator and integrating by parts, one finds

$$P_{\alpha\beta}=-m_\alpha\Gamma_{\alpha\beta}\int d^3\mathbf{v}\, f_\alpha\left[\mathbf{v}\cdot\nabla_\mathbf{v}h_{\alpha\beta}+\left(\frac{m_\beta}{m_\alpha+m_\beta}\right)h_{\alpha\beta}\right]. \tag{18}$$

For isotropic velocity distributions the energy transfer becomes

$$P_{\alpha\beta}=64\pi^3Z_\alpha^2Z_\beta^2e^4\ln\Lambda\int_0^\infty dv v^2 f_\alpha$$
$$\times\left[\frac{1}{m_\beta}\frac{1}{v}\int_0^v du\, f_\beta(u)u^2-\frac{1}{m_\alpha}\int_v^\infty du\, f_\beta(u)u\right]. \tag{19}$$

C. Equilibrium distribution functions

With the aid of Eq. (14), the collision operator between two species may be rewritten as

$$C_{\alpha\beta}=\Gamma_{\alpha\beta}\nabla_\mathbf{v}\cdot\left[\frac{1}{2}(\nabla_\mathbf{v}f_\alpha)\cdot\nabla_\mathbf{v}\nabla_\mathbf{v}g_{\alpha\beta}-\frac{m_\alpha}{m_\alpha+m_\beta}f_\alpha\nabla_\mathbf{v}h_{\alpha\beta}\right]. \tag{20}$$

For isotropic distribution functions the collision operator is

$$C_{\alpha\beta}=\Gamma_{\alpha\beta}\frac{1}{v^2}\frac{\partial}{\partial v}v^2\left[\frac{1}{2}\frac{\partial f_\alpha}{\partial v}\frac{\partial^2 g_{\alpha\beta}}{\partial v^2}-\frac{m_\alpha}{m_\alpha+m_\beta}f_\alpha\frac{\partial h_{\alpha\beta}}{\partial v}\right]$$
$$=\frac{16\pi^2Z_\alpha^2Z_\beta^2e^4\ln\Lambda}{m_\alpha^2}\frac{1}{v^2}\frac{\partial}{\partial v}v^2\left\{\frac{\partial f_\alpha}{\partial v}\frac{1}{3}\right.$$
$$\times\left[\frac{1}{v^3}\int_0^v dv'\, f_\beta(v')v'^4+\int_v^\infty dv'\, f_\beta(v')v'\right]$$
$$\left.+f_\alpha\frac{m_\alpha}{m_\beta}\frac{1}{v^2}\int_0^v dv'\, f_\beta(v')v'^2\right\}. \tag{21}$$

Therefore, the collisional velocity–space flux from Eq. (10) is found to be

$$\mathbf{J}_{\alpha\beta}=-\frac{16\pi^2Z_\alpha^2Z_\beta^2e^4\ln\Lambda}{m_\alpha^2}\left\{\frac{\partial f_\alpha}{\partial v}\frac{1}{3}\right.$$
$$\times\left[\frac{1}{v^3}\int_0^v dv'\, f_\beta(v')v'^4+\int_v^\infty dv'\, f_\beta(v')v'\right]$$
$$\left.+f_\alpha\frac{m_\alpha}{m_\beta}\frac{1}{v^2}\int_0^v dv'\, f_\beta(v')v'^2\right\}\hat{\mathbf{v}}, \tag{22}$$

where $\hat{\mathbf{v}}$ denotes the "radial" direction in velocity space.

Assuming that there are no external forces or spatial gradients, for f_α to be in equilibrium one must have $(\partial f_\alpha/\partial t)_{\text{col.}}=0$. For isotropic velocity distributions, this requirement reduces to $\Sigma_\beta\,\mathbf{J}_{\alpha\beta}=0$, or equivalently

$$\frac{\partial f_\alpha(v)}{\partial v}=-f_\alpha(v)\frac{3\Sigma_\beta[Z_\beta^2(m_\alpha/m_\beta)(1/v^2)\int_0^v dv' f_\beta(v')v'^2]}{[(1/v^3)\int_0^v dv'\,\Sigma_\beta Z_\beta^2 f_\beta(v')v'^4+\int_v^\infty dv'\,\Sigma_\beta Z_\beta^2 f_\beta(v')v']}. \tag{23}$$

For the case of electrons interacting with ions, the electron distribution function will acquire a quasiequilibrium shape while its mean energy is still in the process of changing due to energy exchange with the ion species. Therefore, one may use Eq. (23) to find the electrons' "equilibrium" distribution function f_α, which may then be used in Eq. (19) to arrive at the rate of interspecies energy transfer.

Note that $f_\alpha(v)$ cannot increase with increasing v in any range of velocity space if the distribution is to be held in equilibrium (or quasiequilibrium) solely by collisions with other species (even if those other species have fixed and/or non-Maxwellian distribution functions). Thus one cannot "dig a well" in the electron distribution to cause a radical depletion of the slow-moving electrons which draw energy away from ions, unless one resorts to particle sources and sinks, externally applied electromagnetic fields, transient operating conditions, etc.

IV. ION–ELECTRON HEAT TRANSFER FOR MAXWELLIAN IONS

In this section the general ion–electron heat transfer formulas of the previous section will be applied to the specific case in which the various ion species which are present have Maxwellian velocity distributions.

A. General heat transfer for Maxwellian Ions

For Maxwellian ions with thermal velocity $v_{ti}\equiv\sqrt{2T_i/m_i}$, the distribution function is

$$f_i(v)=\frac{n_i}{\pi^{3/2}v_{ti}^3}\exp\left(-\frac{v^2}{v_{ti}^2}\right). \tag{24}$$

It is assumed for the time being that different ion species in the plasma may have different temperatures.

Substituting Eq. (24) into the expression for the ion–

electron heat transfer, Eq. (19), and integrating by parts, one finds that the power per volume transferred from the ions to the electrons is

$$P_{ie}=16\pi^2e^4\ln\Lambda\int_0^\infty dv\,v^2 f_e(v)\sum_i\frac{Z_i^2n_i}{m_i}\times\left[\frac{2}{\sqrt{\pi}}\frac{m_i}{m_e}\frac{1}{v_{ti}}\exp\left(-\frac{v^2}{v_{ti}^2}\right)-\frac{1}{v}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)\right]. \qquad (25)$$

Now one needs to find the equilibrium electron distribution function $f_e(v)$ to use in Eq. (25) for the heat transfer. By substituting (24) into Eq. (23) and again employing integration by parts, the differential equation determining $f_e(v)$ reduces to

$$\begin{aligned}&\frac{\partial f_e}{\partial v}\left\{\frac{1}{3v}\int_0^v dv'\,f_e(v')v'^4-\frac{v^2}{3}\int_0^v dv'\,f_e(v')v'\right.\\&+\frac{v^2}{3}\int_0^\infty dv'\,f_e(v')v'\\&\left.+\sum_i\frac{Z_i^2n_iv_{ti}^2}{4\pi^{3/2}v}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)-\frac{v}{v_{ti}}\exp\left(-\frac{v^2}{v_{ti}^2}\right)\right]\right\}\\&+f_e(v)\left\{\int_0^v dv'\,f_e(v')v'^2+\sum_i\frac{Z_i^2n_i}{2\pi^{3/2}}\frac{m_e}{m_i}\right.\\&\left.\times\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)-\frac{v}{v_{ti}}\exp\left(-\frac{v^2}{v_{ti}^2}\right)\right]\right\}=0\end{aligned} \qquad (26)$$

for the case of electrons in the presence of multiple Maxwellian ion species.

If Eq. (26) is solved numerically and its solution for the equilibrium $f_e(v)$ used with Eq. (25), one will find the exact value for the heat transfer to electrons from Maxwellian ions for any choice of parameters. However, to obtain useful analytical expressions and simplified numerical results, further approximations are required.

One should also note that by using Eq. (25) and assuming that the electrons remain perfectly Maxwellian (and allowing the ratio of ion and electron temperatures to remain arbitrary), the result first found by Spitzer[3,4] may be obtained:

$$(P_{ie})_{\text{Spitzer}}=\frac{4\sqrt{2\pi m_im_e}Z_i^2e^4n_in_e\ln\Lambda}{(m_iT_e+m_eT_i)^{3/2}}(T_i-T_e). \qquad (27)$$

This classical Spitzer energy transfer rate will serve as a useful basis for comparison with the modified rate described by Eqs. (25) and (26).

B. Modification of Spitzer ion–electron heat transfer

If the electrons moving more slowly than the ions are partially depleted due to energy upscattering from the ions, the heat transfer rate will be less than the Spitzer result. To examine this effect, it will be assumed that the ions are Maxwellian and are moving significantly more slowly that the electrons, but the electron distribution will not be assumed to be Maxwellian. This calculation will produce a modification factor to the Spitzer heat transfer rate which will reduce to the answer obtained by Rosenbluth[1,2] in the proper limit.

1. Relationship between slow electron depletion and reduction of ion–electron heat transfer

Before proceeding with the main line of the derivation, one of the key arguments used in the more intuitive analysis of Sec. II will now be confirmed; in particular, it will be shown that the ion–electron heat transfer rate is essentially proportional to the number of electrons moving more slowly than the ions, or in other words approximately proportional to $f_e(v=0)$.

For $v_{te}\gg v_{ti}$ one may assume that the electron distribution shape is governed by electrons with velocities v such that $v\gg v_{ti}$; therefore, Eq. (25) becomes

$$P_{ie}\approx\frac{16\pi^2Z_i^2e^4n_i\ln\Lambda}{m_i}\left[\frac{T_i}{m_e}f_e(0)-\int_0^\infty dv\,f_e(v)v\right]. \qquad (28)$$

For Maxwellian electrons Eq. (28) reduces to

$$P_{ie}=(P_{ie})_{\text{Spitzer}}=\frac{16\pi^2Z_i^2e^4n_i\ln\Lambda}{m_im_e}(T_i-T_e)\times[f_e(0)]_{\text{Maxwellian}}. \qquad (29)$$

Equation (29) is clearly the $v_{te}\gg v_{ti}$ limiting form of the full Spitzer result of Eq. (27). Assuming that the electrons do not deviate too much from a Maxwellian distribution, then one may use

$$\int_0^\infty dv\,v\,f_e(v)\approx\frac{T_e}{m_e}f_e(0) \qquad (30)$$

in Eq. (28). Dividing the resulting expression by Eq. (29) produces the result

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}}\approx\frac{(T_i/m_e)f_e(0)-(T_e/m_e)f_e(0)}{[(T_i-T_e)/m_e][f_e(0)]_{\text{Maxwellian}}}=\frac{f_e(0)}{[f_e(0)]_{\text{Maxwellian}}}. \qquad (31)$$

Because substantially non-Maxwellian electron distributions will arise only when $T_i\gg T_e$ (causing interactions with ions to interfere strongly with the electron distribution), the correction to the Spitzer rate will reduce to (31) even when the assumption underlying Eq. (30) breaks down:

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}}\approx\frac{(T_i/m_e)f_e(0)}{(T_i/m_e)[f_e(0)]_{\text{Maxwellian}}}=\frac{f_e(0)}{[f_e(0)]_{\text{Maxwellian}}}. \qquad (32)$$

Therefore, when the electron distribution function is altered so that fewer than the Maxwellian number of electrons have very slow speeds, the heat transfer rate is reduced accordingly.

2. Derivation of electron distribution and heat transfer

Consider electrons with velocity v such that $v_{ti} \ll v \ll v_{te}$. In this case, one may make the approximations $\exp(-v^2/v_{ti}^2) \to 0$ and $\mathrm{erf}(v/v_{ti}) \to 1$ in Eq. (26).

Using these approximations, the differential equation for the electron distribution becomes

$$\frac{\partial f_e}{\partial v}\left[\sum_i \frac{Z_i^2 n_i}{4\pi v^3}\frac{T_i}{m_i}+\frac{1}{3}\int_0^\infty dv'\, f_e(v')v'\right] +f_e(v)\,\frac{1}{v^2}\left[\frac{v^3}{3}f_e(0)+\sum_i \frac{Z_i^2 n_i}{4\pi}\frac{m_e}{m_i}\right]=0. \quad (33)$$

Assuming that the electrons are nearly Maxwellian so that Maxwellian values may be used for the electron-related quantities within the brackets, one obtains

$$\frac{\partial f_e}{\partial v}\frac{T_e}{m_e}\left[v^3+3\sqrt{\frac{\pi}{2}}\sum_i \frac{Z_i^2 n_i}{n_e}\frac{T_i}{m_i}\sqrt{\frac{T_e}{m_e}}\right] +v f_e\left[v^3+3\sqrt{\frac{\pi}{2}}\sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\left(\frac{T_e}{m_e}\right)^{3/2}\right]=0. \quad (34)$$

The form of Eq. (34) suggests that one define a critical velocity v_c for the electrons as

$$v_c^3 \equiv 3\sqrt{\frac{\pi}{2}}\sum_i \frac{Z_i^2 n_i}{n_e}\frac{T_i}{m_i}\sqrt{\frac{T_e}{m_e}} =\frac{3\sqrt{\pi}}{4}\sum_i \frac{Z_i^2 n_i}{n_e} v_{ti}^2 v_{te}. \quad (35)$$

This definition is the same critical velocity which was found in the introductory section. By using the critical velocity and assuming that all of the ion species are at the same temperature T_i, Eq. (34) may be solved to find $f_e(v)$:

$$f_e(v)=f_e(0)\exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv'\, v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\right\}. \quad (36)$$

One may find $f_e(0)$ from the normalization condition in Eq. (9). It should be realized that the derivation of this distribution function assumed that $v_{ti} \ll v \ll v_{te}$.

The integral in the exponent may be evaluated:[9]

$$\begin{aligned}\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)} &= \frac{1}{2}v^2-\left(1-\frac{T_e}{T_i}\right)v_c^3\int_0^v \frac{dv' v'}{(v'^3+v_c^3)}\\ &= \frac{1}{2}v^2+\left(1-\frac{T_e}{T_i}\right)\frac{v_c^2}{3}\left\{\frac{1}{2}\ln\left[\frac{(v+v_c)^2}{v^2-vv_c+v_c^2}\right]\right.\\ &\quad \left.-\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\sqrt{3}\tan^{-1}\left(\frac{2v-v_c}{\sqrt{3}v_c}\right)\right\}. \quad (37)\end{aligned}$$

It is clear from Eq. (37) that in the classical limit ($v_c \to 0$) the distribution function becomes the usual Maxwellian.

Now the electron distribution function found above may be used in the expression from Eq. (25) for ion–electron heat transfer in the presence of Maxwellian ion species. Making this substitution and dividing by the Spitzer energy transfer rate from Eq. (27), one obtains

$$\begin{aligned}\frac{P_{ie}}{(P_{ie})_{\mathrm{Spitzer}}} \approx{}& \frac{\sqrt{\pi}}{2}\frac{T_e}{(T_i-T_e)}\left(1+\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{3/2}\biggl\{\int_0^\infty dv v^2 \exp\biggl[-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\biggr]\\ &\times\biggl[\frac{2}{\sqrt{\pi}}\left(\frac{m_i}{m_e}\right)^{3/2}\sqrt{\frac{T_e}{T_i}}\exp\left(\frac{-v^2}{v_{ti}^2}\right)-\frac{v_{te}}{v}\,\mathrm{erf}\left(\frac{v}{v_{ti}}\right)\biggr]\biggr]\biggr\}\biggl[\int_0^\infty dv v^2\exp\biggl\{-\frac{m_e}{T_e}\int_0^v\frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\biggr\}\biggr]^{-1}. \quad (38)\end{aligned}$$

Note that $v \sim v_{ti}$ corrections have been retained so that the correct Spitzer rate will be recovered for $v_c^3 \to 0$.

3. Useful approximate answer

A simplified answer can be extracted from Eq. (38) by analytical means.

In the first integral of Eq. (38), the integrand is appreciable only for v of the order of v_{ti} or smaller, so one may assume that v and v' are of the order v_{ti} and thus much smaller than v_c. In this limit the integral becomes

$$\begin{aligned}&\int_0^\infty dv v^2\exp\left[-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\right]\frac{2}{\sqrt{\pi}}\\ &\quad\times\left(\frac{m_i}{m_e}\right)^{3/2}\sqrt{\frac{T_e}{T_i}}\exp\left(\frac{-v^2}{v_{ti}^2}\right)\\ &\approx\int_0^\infty dv v^2\exp\left[-\left(1+\frac{m_e}{m_i}\right)\frac{v^2}{v_{ti}^2}\right]\frac{2}{\sqrt{\pi}}\left(\frac{m_i}{m_e}\right)^{3/2}\sqrt{\frac{T_e}{T_i}}\\ &\approx v_{te}\frac{T_i}{m_e}. \quad (39)\end{aligned}$$

The integrands of the remaining integrals in Eq. (38) are not restricted to the $v \sim v_{ti}$ velocity range [they do not have the

$\exp(-v^2/v_{ti}^2)$ term], so in general the electron velocity v in these integrals extends to the order of v_{te}, or much larger than v_c. In this limit, the integral in the exponentials of these terms may be approximated by using Eq. (37), so that

$$\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)} \approx \frac{1}{2}v^2 - \left(1-\frac{T_e}{T_i}\right)v_c^2\,\frac{2\pi}{3\sqrt{3}}. \tag{40}$$

Therefore the remaining integrals in Eq. (38) may be approximated as

$$\begin{aligned}
&\int_0^\infty dv\, v\, v_{te}\, \mathrm{erf}\left(\frac{v}{v_{ti}}\right) \\
&\quad\times \exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\right\} \\
&\quad\approx v_{te}\exp\left\{\frac{2\pi}{3\sqrt{3}}\left(1-\frac{T_e}{T_i}\right)\frac{m_e v_c^2}{T_e}\right\}\int_0^\infty dv\, v \\
&\quad\times \exp\left\{-\frac{m_e v^2}{2T_e}\right\} \\
&\quad\approx v_{te}\frac{T_e}{m_e}\exp\left\{\frac{2\pi}{3\sqrt{3}}\left(1-\frac{T_e}{T_i}\right)\frac{m_e v_c^2}{T_e}\right\};
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\int_0^\infty dv\, v^2 \exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3+(T_e/T_i)v_c^3)}{(v'^3+v_c^3)}\right\} \\
&\quad\approx \exp\left\{\frac{2\pi}{3\sqrt{3}}\left(1-\frac{T_e}{T_i}\right)\frac{m_e v_c^2}{T_e}\right\}\int_0^\infty dv\, v^2 \exp\left\{-\frac{m_e v^2}{2T_e}\right\} \\
&\quad\approx \sqrt{\frac{\pi}{2}}\left(\frac{T_e}{m_e}\right)^{3/2}\exp\left\{\frac{2\pi}{3\sqrt{3}}\left(1-\frac{T_e}{T_i}\right)\frac{m_e v_c^2}{T_e}\right\}.
\end{aligned} \tag{42}$$

Using these approximations, Eq. (38) becomes

$$\begin{aligned}
\frac{P_{ie}}{(P_{ie})_{\mathrm{Spitzer}}} &\approx \left(1+\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{3/2} \\
&\quad\times \exp\left\{-\left(\frac{2\pi^2}{3^{5/4}}\sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{2/3}\right\},
\end{aligned} \tag{43}$$

where some corrections of order v_{ti}^2/v_{te}^2 have been neglected in the asymptotic evaluation of the integrals.

For the case in which only one ion species is present ($Z_i n_i = n_e$) and the temperature ratio T_i/T_e remains moderate, this expression clearly reduces to precisely the answer obtained by Rosenbluth:[2]

$$\frac{(P_{ie})_{\mathrm{Rosenbluth}}}{(P_{ie})_{\mathrm{Spitzer}}} \approx 1 - \left(\frac{2\pi^2}{3^{5/4}}Z_i\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{2/3}. \tag{44}$$

It is useful to realize that $2\pi^2/3^{5/4} \approx 5.000$.

Now the significance of this work may be seen. While Rosenbluth's answer is just an expansion valid for T_i not much larger than T_e (and indeed takes on a nonphysical negative value if one chooses T_i/T_e to be sufficiently large), the result presented in Eq. (38) and even the more approximate one of Eq. (43) are considerably more accurate, and they give sensible answers even for large T_i/T_e. The accuracy of Eq. (38) will next be demonstrated by numerically integrating this expression and comparing the result with the output of a Fokker–Planck code for a wide range of T_i/T_e values.

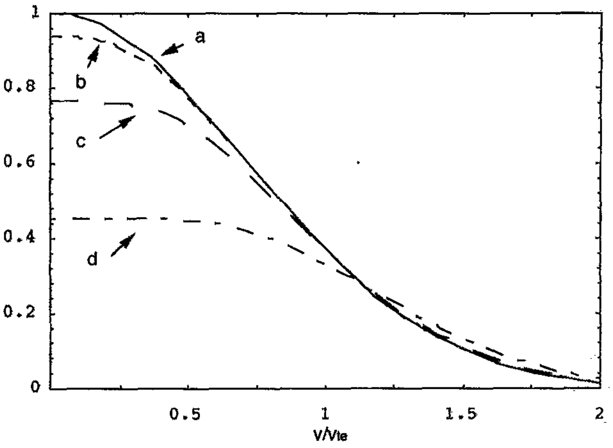

FIG. 1. Electron distribution for pure hydrogen (^{1}H) plasma with (a) $T_i/T_e = 1$, (b) 10, (c) 100, and (d) 1000.

4. More accurate answer via numerical integration

Mathematica[10] has been used to plot the normalized distribution function from Eq. (36) for various values of T_i/T_e (with $Z_i \equiv 1$ and $A \equiv 1$ for all of the curves). Figure 1 shows the plots for $T_i/T_e = 1$, 10, 100, and 1000. As may be seen in the figure, the flattening of the electron distribution at small velocities becomes more pronounced as the temperature ratio increases, as expected. [Some of the approximations made in obtaining Eq. (36) begin to break down for $T_i/T_e = 1000$, but the general appearance of the distribution function at these parameters is still highly revealing.]

The correction to the Spitzer rate as described by Eq. (38) has been calculated via numerical integration with Mathematica. The resulting graphs are shown in Figs. 2–4 for the cases in which the plasma consists of pure light hydrogen, pure deuterium, and pure helium-3. These results for the case of Maxwellian ions are contrasted in the graphs with the results for the case of monoenergetic ions, which will be derived in the next section.

As may be seen in the graphs, the correction factor begins to level off for large T_i/T_e. This behavior is to be expected, for if one continues to hold the ion distribution perfectly Maxwellian and redefines T_e to be 2/3 of the mean electron energy (even when the electron distribution becomes non-Maxwellian), the ion–electron heat transfer should return toward the $T_i/T_e \to \infty$ Spitzer rate for extremely large values of T_i/T_e (when $v_{ti}^2 \gg v_{te}^2$, so the ion velocity is the dominant determinant of the relative collision velocity). At $T_i/T_e = 1000$, this upward return back toward the Spitzer formula has not yet begun [except for the case of light hydrogen with monoenergetic ions, as shown by curve (a) of Fig. 2], but the correction factor is beginning to level off in preparation for the upward turn.

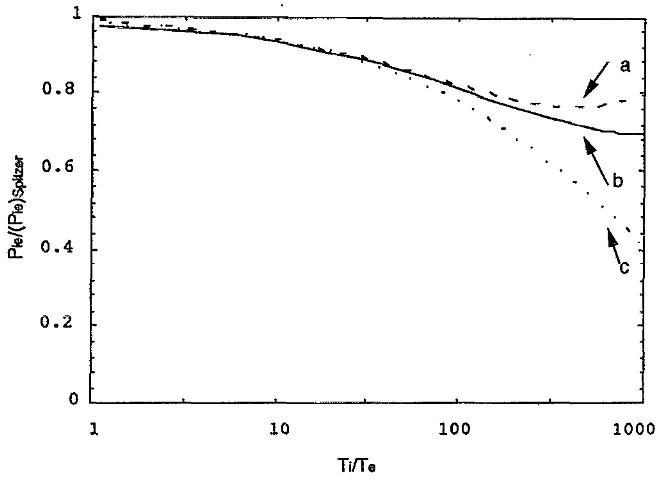

FIG. 2. Correction factor to Spitzer ion-electron energy transfer rate for pure hydrogen (^{1}H) plasma as a function of T_i/T_e: (a) monoenergetic ions, (b) Maxwellian ions, (c) approximate answer from Eq. (45).

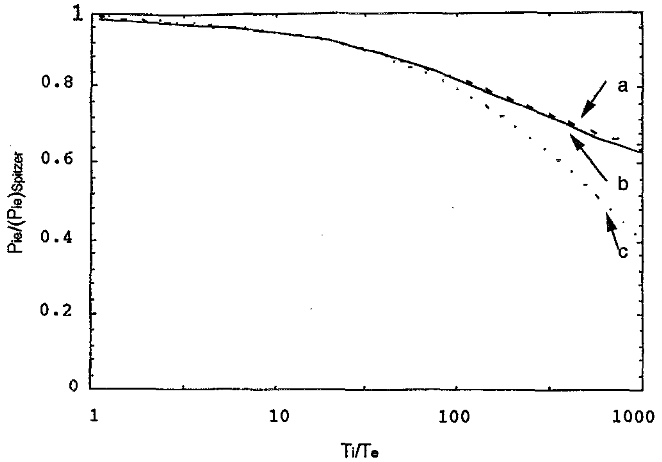

FIG. 4. Correction factor to Spitzer ion–electron energy transfer rate for pure helium-3 plasma as a function of T_i/T_e: (a) monoenergetic ions, (b) Maxwellian ions, (c) approximate answer from Eq. (45).

Along with the plots based on Eq. (38), Figs. 2–4 also present graphs of the more approximate but more readily usable answer,

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}} \approx \left(1+\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{3/2} \times \exp\left\{-\left(3.5\sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{2/3}\right\}. \tag{45}$$

Note that the coefficient in the exponent has been changed from the previous approximate value of 5.00 to the present value of 3.5 in order to match the complete results more accurately over a wider range of values of T_i/T_e. As one may see in the graphs, this approximate answer matches the full analytical results quite well for temperature ratios such that

$$1 \leqslant \sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_p}{m_i}\frac{T_i}{T_e} \leqslant 50, \tag{46}$$

in which m_p is the proton mass.

Figure 5 again shows the numerically integrated result for the case of deuterium with a Maxwellian ion distribution, but now that curve is compared with the results obtained by Galambos[11,12] using the FPPAC Fokker–Planck code.[13,14] It may be seen that there is fairly good agreement between the present analytical results and the code results for the heat transfer rate. Methods for obtaining even more precise analytical expressions for the energy exchange rate are discussed in Appendix B.

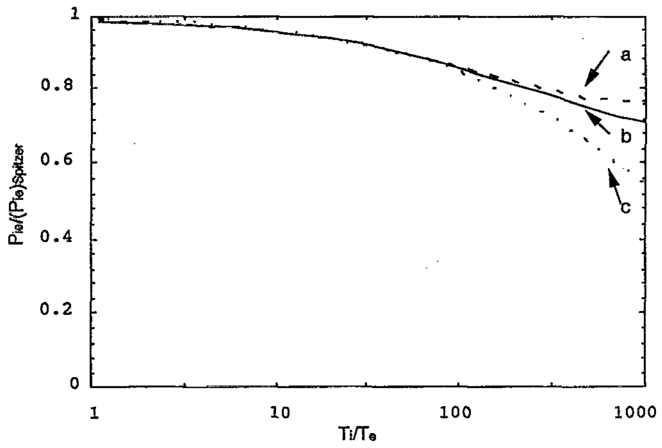

FIG. 3. Correction factor to Spitzer ion–electron energy transfer rate for pure deuterium plasma as a function of T_i/T_e: (a) monoenergetic ions, (b) Maxwellian ions, (c) approximate answer from Eq. (45).

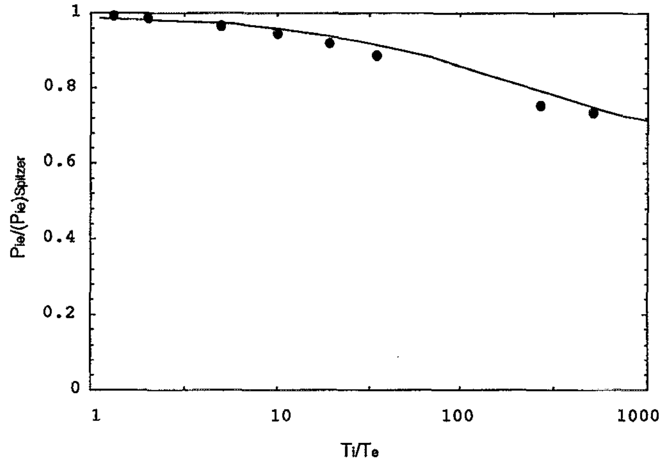

FIG. 5. Comparison of analytical result from Eq. (38) (line) with code results (points) from[11,12] for deuterium with Maxwellian ion distribution.

V. ION–ELECTRON HEAT TRANSFER FOR MONOENERGETIC IONS

Now the heat transfer will be calculated assuming that the ions all have velocity v_i, or energy $E_i = m_i v_i^2/2$. This calculation is relevant to the evaluation of fusion concepts such as those proposed by Bussard[5] and Maglich,[6] which are intended to operate with nearly monoenergetic ion beams that have energies much greater than the mean electron energy. (In spherically convergent systems of the type proposed by Bussard, the density generally varies as roughly $1/r^2$, where r is the radial distance from the center of the spherical plasma.[5] Therefore most of the collisions occur in the dense central region, where particles are coming from and returning to all directions, and so the assumption of isotropy made in the present calculations is valid. Anisotropy could be a more serious concern in Maglich's migma configuration,[6] although the present isotropic calculation may be considered a first-order treatment of the plasma behavior in that device.)

A. Derivation of electron distribution and heat transfer

For isotropic but monoenergetic ions, the distribution function is

$$f_i(v) = \frac{n_i}{4\pi v_i^2}\,\delta(v - v_i). \tag{47}$$

By substituting this distribution function in Eq. (19), the power per volume transferred from the ions to the electrons is found to be

$$P_{ie} = 16\pi^2 Z_i^2 e^4 n_i \ln\Lambda \left[\frac{1}{m_e v_i}\int_0^{v_i} dv'\, f_e(v')v'^2 - \frac{1}{m_i}\int_{v_i}^{\infty} dv'\, f_e(v')v'\right]. \tag{48}$$

Using the monoenergetic ion distribution together with the earlier general formula for the equilibrium electron distribution function, Eq. (23), one obtains

$$\begin{aligned}\frac{\partial f_e}{\partial v}\frac{1}{3}&\left[\frac{1}{v^3}\int_0^{v} dv'\, f_e(v')v'^4 + \sum_i \frac{Z_i^2 n_i}{4\pi v^3}\, v_i^2 \Theta(v - v_i)\right.\\ &\left. + \int_v^{\infty} dv'\, f_e(v')v' + \sum_i \frac{Z_i^2 n_i}{4\pi v_i}\,\Theta(v_i - v)\right] + f_e(v)\\ &\times\left[\frac{1}{v^2}\int_0^{v} dv'\, f_e(v')v'^2 + \sum_i \frac{Z_i^2 n_i}{4\pi}\frac{m_e}{m_i}\frac{1}{v^2}\,\Theta(v - v_i)\right]\\ &= 0.\end{aligned} \tag{49}$$

For v_i substantially smaller than v_{te}, the electron distribution will be governed by the equation for the overwhelming majority of the electrons which have $v > v_i$, so one may set $\Theta(v - v_i) = 1$ and $\Theta(v_i - v) = 0$ in Eq. (49) in order to find a good expression for $f_e(v)$. However, if v_i is comparable to v_{te}, phenomena occurring on both sides of $v = v_i$ must be taken into account.

B. Useful approximate answer

For electrons with $v_{te} \gg v > v_i$, Eq. (49) may be approximated by

$$\frac{\partial f_e}{\partial v}\left[\sum_i \frac{Z_i^2 n_i}{4\pi v^3}\frac{v_i^2}{3} + \frac{1}{3}\int_0^{\infty} dv'\, f_e(v')v'\right] + f_e\frac{1}{v^2}\left[\frac{v^3}{3}f_e(0) + \sum_i \frac{Z_i^2 n_i}{4\pi}\frac{m_e}{m_i}\right] = 0. \tag{50}$$

Note that this equation for electrons interacting with monoenergetic ions is exactly the same as Eq. (33) for electrons interacting with Maxwellian ions in the corresponding velocity range ($v_{te} \gg v \gg v_{ti}$), provided that one uses $v_{ti}^2 \rightarrow 2v_i^2/3$, or $T_i \rightarrow 2E_i/3$.

Accordingly, the critical velocity for the electrons is now defined as

$$v_c^3 \equiv \sqrt{2\pi}\sum_i \frac{Z_i^2 n_i}{n_e}\frac{E_i}{m_i}\sqrt{\frac{T_e}{m_e}}. \tag{51}$$

Similarly the electron distribution function is

$$\begin{aligned}f_e(v) = \frac{n_e}{4\pi}\exp&\left\{-\frac{m_e}{T_e}\int_0^{v}\frac{dv'v'(v'^3 + \frac{3}{2}(T_e/E_i)v_c^3)}{(v'^3 + v_c^3)}\right\}\\ &\times\left[\int_0^{\infty} dv'v'^2\right.\\ &\left.\times\exp\left\{-\frac{m_e}{T_e}\int_0^{v'}\frac{dv''v''(v''^3 + \frac{3}{2}(T_e/E_i)v_c^3)}{(v''^3 + v_c^3)}\right\}\right]^{-1}.\end{aligned} \tag{52}$$

The power density transferred from the ions to the electrons may be approximated as

$$P_{ie} \approx \frac{16\pi^2 Z_i^2 e^4 n_i \ln\Lambda}{m_i}\left[\frac{2}{3}\frac{E_i}{m_e}f_e(0) - \int_0^{\infty} dv\, f_e(v)v\right]. \tag{53}$$

This expression is identical to Eq. (28) provided that one again makes the identification $T_i \rightarrow 2E_i/3$. Because of the exact correspondence between Eqs. (50) and (53) and their predecessors in the Maxwellian ion case, the Maxwellian results may be used here, provided the proper substitution is made for the ion temperature in each case.

By analogy with the earlier Maxwellian results, a useful approximation for the heat transfer is (taking the numerical coefficient in the exponential to be $3.5 \cdot 2/3 \approx 2.4$)

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}} \approx \left(1 + \frac{2}{3}\frac{m_e}{m_i}\frac{E_i}{T_e}\right)^{3/2} \times\exp\left\{-\left(2.4\sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\frac{E_i}{T_e}\right)^{2/3}\right\}. \tag{54}$$

C. More accurate answer

By using the electron distribution function of Eq. (52) in Eq. (48) and dividing by the Spitzer rate, a more accurate expression for the correction factor is found to be

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}} \approx \sqrt{\frac{\pi}{2}} \frac{T_e}{(\frac{2}{3}E_i - T_e)} \left(1 + \frac{2}{3}\frac{m_e}{m_i}\frac{E_i}{T_e}\right)^{3/2} \left[\frac{m_i}{m_e}\frac{1}{v_i}\int_0^{v_i} dv v^2 \exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3 + \frac{3}{2}(T_e/E_i)v_c^3)}{(v'^3+v_c^3)}\right\} - \int_{v_i}^{\infty} dv v \right.$$
$$\left. \times \exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3 + \frac{3}{2}(T_e/E_i)v_c^3)}{(v'^3+v_c^3)}\right\}\right]\left[\sqrt{\frac{m_e}{T_e}}\int_0^{\infty} dv v^2 \exp\left\{-\frac{m_e}{T_e}\int_0^v \frac{dv' v'(v'^3 + \frac{3}{2}(T_e/E_i)v_c^3)}{(v'^3+v_c^3)}\right\}\right]^{-1}. \tag{55}$$

Mathematica was again employed in order to numerically integrate and graph this improved expression for the ion–electron heat transfer rate in the case monoenergetic ions. The results are shown in Figs. 2–4 (along with the results for the Maxwellian ion case) for plasmas consisting of pure light hydrogen, pure deuterium, and pure ^{3}He, respectively. In the graphs, the effective ion temperature has been defined as $T_i \equiv 2E_i/3$.

Since the most important feature about the interactions of the ions with the electrons is that the ion speeds are typically much smaller than the electron thermal speed, one would expect that the heat transfer rate would depend only on the mean ion energy and not the particular ion distribution shape (except at very large temperature ratios, $T_i/T_e \sim 1000$, when the mean ion and electron speeds start to become comparable). This behavior is indeed quite evident in Figs. 2–4.

Based on the comparison with the analytical and code results for Maxwellian ions, this monoenergetic ion answer appears to be fairly accurate. However, techniques for obtaining an even more precise analytical answer for the monoenergetic ion case are discussed in Appendix B.

VI. CONCLUSIONS

Corrections to the classical Spitzer rate of ion–electron energy exchange were calculated for the case of large T_i/T_e ratios. The results of these calculations are substantially more accurate and more broadly applicable than the original result of Rosenbluth.[2]

A useful expression for the correction factor is

$$\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}} \approx \left(1 + \frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{3/2} \times \exp\left\{-\left(3.5\sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{2/3}\right\}. \tag{56}$$

This result assumes that all of the ion species are Maxwellian and at the same temperature T_i. If the ions are non-Maxwellian, an effective ion temperature for use in the above equation may be defined in terms of the mean ion energy, $T_i \equiv 2\langle E_i \rangle/3$. Note that this simple approximation yields accurate results only for the temperature range

$$1 \leqslant \sum_i \frac{Z_i^2 n_i}{n_e}\frac{m_p}{m_i}\frac{T_i}{T_e} < 50. \tag{57}$$

For temperature ratios larger than this range, the approximate answer given above begins to underestimate the actual heat transfer, so in such cases one should use the results of one of the more sophisticated calculations presented in the paper.

These more accurate analytical expressions for the correction factor were numerically integrated and graphed using Mathematica, and the results were summarized in graphs for plasmas of various compositions. The results generally agree with those obtained by Galambos[11,12] with a Fokker–Planck code.

As presented in Appendix B, iterative methods may be employed if one desires to obtain even more accurate analytical expressions for the correction factor for the two cases of Maxwellian ions and monoenergetic ions.

The correction factor derived in this paper may be incorporated into calculations of electron energy balance and bremsstrahlung radiation in order to improve the accuracy of those calculations.[15]

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APPENDIX A: FOKKER–PLANCK CALCULATIONS FOR GENERAL ISOTROPIC VELOCITY DISTRIBUTIONS

1. Rosenbluth potentials

For isotropic velocity distributions, the Rosenbluth potentials (12) and (13) may be integrated over all angles in velocity space as follows (θ is the angle between $\mathbf{u}$ and $\mathbf{v}$):

$$
\begin{aligned}
h_{\alpha\beta}(\mathbf{v}) &= 2\pi \, \frac{m_\alpha + m_\beta}{m_\beta} \int \frac{f_\beta(u) u^2 \sin\theta \, du \, d\theta}{\sqrt{u^2+v^2-2uv\cos\theta}} \\
&= 2\pi \, \frac{m_\alpha + m_\beta}{m_\beta} \\
&\quad \times \left[\int \frac{du u^2 f_\beta(u) \sqrt{u^2+v^2-2uv\cos\theta}}{uv} \right]_0^\pi \\
&= \frac{m_\alpha + m_\beta}{m_\beta} \int_0^\infty (du \; 4\pi u^2) f_\beta(u) \\
&\quad \times \left[\frac{u\Theta(v-u) + v\Theta(u-v)}{uv} \right],
\end{aligned}
\tag{A1}
$$

in which $\Theta(x)$ is the unit step function.

Similarly, one finds that

$$
\begin{aligned}
g_{\alpha\beta}(\mathbf{v}) &= 2\pi \int f_\beta(u) \sqrt{u^2+v^2-2uv\cos\theta} \; u^2 \sin\theta \, du \, d\theta \\
&= 2\pi \left[\int \frac{du u^2 \, f_\beta(u) (u^2+v^2-2uv\cos\theta)^{3/2}}{3uv} \right]_0^\pi \\
&= \frac{1}{3} \int_0^\infty (du \; 4\pi u^2) f_\beta(u) \\
&\quad \times \left[\frac{u(u^2+3v^2)\Theta(v-u) + v(v^2+3u^2)\Theta(u-v)}{uv} \right].
\end{aligned}
\tag{A2}
$$

The following derivatives of the Rosenbluth potentials are also needed for the calculations presented in this paper:

$$
\frac{\partial h_{\alpha\beta}}{\partial v} = -4\pi \left(\frac{m_\alpha + m_\beta}{m_\beta} \right) \frac{1}{v^2} \int_0^v du \; f_\beta(u) u^2;
\tag{A3}
$$

$$
\begin{aligned}
\frac{\partial^2 g_{\alpha\beta}}{\partial v^2} &= \frac{\partial}{\partial v} \frac{4\pi}{3} \left[\int_0^v du \; f_\beta(u) \left(3u^2 - \frac{u^4}{v^2} - 2uv \right) \right. \\
&\quad \left. + 2v \int_0^\infty du \; f_\beta(u) u \right] \\
&= \frac{8\pi}{3} \left[\frac{1}{v^3} \int_0^v du \; f_\beta(u) u^4 + \int_v^\infty du \; f_\beta(u) u \right].
\end{aligned}
\tag{A4}
$$

2. Collision operators

The collision operator between two species which was given in Eq. (21) may not seem immediately familiar, so it will now be explicitly shown that this expression for the collision operator reduces to a previously published result. Calculating the divergence in Eq. (21), one finds

$$
\begin{aligned}
C_{\alpha\beta} &= \frac{16\pi^2 e^4 \ln \Lambda Z_\alpha^2 Z_\beta^2}{m_\alpha^2 v^2} \frac{\partial}{\partial v} \left(v^2 \left\{ \frac{\partial f_\alpha}{\partial v} \frac{1}{3} \left[\frac{1}{v^3} \int_0^v du \; f_\beta(u) u^4 + \int_v^\infty du \; f_\beta(u) u \right] + f_\alpha \frac{m_\alpha}{m_\beta} \frac{1}{v^2} \int_0^v du \; f_\beta(u) u^2 \right\} \right) \\
&= \frac{16\pi^2 e^4 \ln \Lambda Z_\alpha^2 Z_\beta^2}{m_\alpha^2 v^2} \frac{\partial}{\partial v} \left\{ \frac{\partial f_\alpha}{\partial v} \frac{1}{3} \left[\frac{1}{v} \int_0^v du \; f_\beta(u) u^4 + v^2 \int_0^\infty du \; f_\beta(u) u - v^2 \int_0^v du \; f_\beta(u) u \right] + f_\alpha \frac{m_\alpha}{m_\beta} \int_0^v du \; f_\beta(u) u^2 \right\} \\
&= \frac{16\pi^2 e^4 \ln \Lambda Z_\alpha^2 Z_\beta^2}{3 m_\alpha^2} \left\{ \frac{\partial^2 f_\alpha}{\partial v^2} \left[\frac{1}{v^3} \int_0^v du \; f_\beta(u) u^4 + \int_v^\infty du \; f_\beta(u) u \right] \right. \\
&\quad \left. + \frac{\partial f_\alpha}{\partial v} \left[\int_0^v du \; f_\beta(u) \left(3 \frac{m_\alpha}{m_\beta} \frac{u^2}{v^2} - \frac{u^4}{v^4} \right) + \frac{2}{v} \int_v^\infty du \; f_\beta(u) u \right] + 3 \frac{m_\alpha}{m_\beta} f_\alpha(v) f_\beta(v) \right\}.
\end{aligned}
\tag{A5}
$$

For like particles the collision operator becomes

$$
\begin{aligned}
C_{\alpha\alpha} &= \frac{16\pi^2 e^4 \ln \Lambda Z_\alpha^4}{3 m_\alpha^2} \left\{ \frac{\partial^2 f_\alpha}{\partial v^2} \left[\frac{1}{v^3} \int_0^v du \; f_\alpha(u) u^4 + \int_v^\infty du \; f_\alpha(u) u \right] \right. \\
&\quad \left. + \frac{\partial f_\alpha}{\partial v} \left[\int_0^v du \; f_\alpha(u) \left(3 \frac{u^2}{v^2} - \frac{u^4}{v^4} \right) + \frac{2}{v} \int_v^\infty du \; f_\alpha(u) u \right] + 3[f_\alpha(v)]^2 \right\} \\
&= \frac{8\pi^2 e^4 \ln \Lambda Z_\alpha^4}{m_\alpha^2} \left\{ \frac{2}{3} \frac{\partial^2 f_\alpha}{\partial v^2} \left[\frac{1}{v^3} \int_0^v du \; f_\alpha(u) u^4 + \int_v^\infty du \; f_\alpha(u) u \right] + 2[f_\alpha(v)]^2 \right. \\
&\quad \left. + \frac{4}{3v} \frac{\partial f_\alpha}{\partial v} \left[\int_0^\infty du \; f_\alpha(u) u - \int_0^v du \; f_\alpha(u) u \left(1 - \frac{u}{v} \right)^2 \left(1 + \frac{u}{2v} \right) \right] \right\}.
\end{aligned}
\tag{A6}
$$

This last expression for $C_{\alpha\alpha}$ matches Eq. (1) of Ref. 8.

APPENDIX B: EVEN MORE ACCURATE (BUT NASTIER) HEAT TRANSFER EXPRESSIONS

1. Improved answer for Maxwellian ion case

It should be possible to obtain an even more accurate answer by returning to the full nonlinear first-order differential equation for the electron distribution function in the presence of Maxwellian ions, Eq. (26). This first-order equation for f_e may be iterated, so that the coefficients are found by using a less accurate expression for the distribution, which will be denoted f_e^*:

$$\begin{aligned}&\frac{\partial f_e}{\partial v}\Biggl\{\frac{1}{3v}\int_0^v dv'\, f_e^*(v')v'^4+\frac{v^2}{3}\int_v^\infty dv'\, f_e^*(v')v'+\sum_i \frac{Z_i^2 n_i v_{ti}^2}{4\pi^{3/2}v}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)-\frac{v}{v_{ti}}\exp\left(-\frac{v^2}{v_{ti}^2}\right)\biggr]\Biggr\}\\&\quad+f_e(v)\Biggl\{\int_0^v dv'\, f_e^*(v')v'^2+\sum_i \frac{Z_i^2 n_i}{2\pi^{3/2}}\frac{m_e}{m_i}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)-\frac{v}{v_{ti}}\exp\left(-\frac{v^2}{v_{ti}^2}\right)\biggr]\Biggr\}=0.\end{aligned}\tag{B1}$$

Solving this equation, the iterated solution for the distribution function expressed in terms of the previous iteration's solution is

$$\begin{aligned}f_e(v)=f_e(0)\exp\Biggl\{&-\int_0^v\Biggl[dv'v'\Biggl\{3\int_0^{v'} dv''\, f_e^*(v'')v''^2+\sum_i \frac{3Z_i^2 n_i}{2\pi^{3/2}}\frac{m_e}{m_i}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{ti}}\right)-\frac{v'}{v_{ti}}\exp\left(-\frac{v'^2}{v_{ti}^2}\right)\biggr]\Biggr\}\\&\times\Biggl\{\int_0^{v'} dv''\, f_e^*(v'')v''^4+v'^3\int_{v'}^\infty dv''\, f_e^*(v'')v''+\sum_i \frac{3Z_i^2 n_i v_{ti}^2}{4\pi^{3/2}}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{ti}}\right)-\frac{v'}{v_{ti}}\exp\left(-\frac{v'^2}{v_{ti}^2}\right)\biggr]\Biggr\}^{-1}\Biggr]\Biggr\}.\end{aligned}\tag{B2}$$

One may then find $f_e(0)$ directly from the normalization condition as usual.

If one begins the iteration process by assuming that $f_e^*(v)$ is Maxwellian and characterized by the thermal velocity v_{te}, then the distribution function of Eq. (B2) becomes

$$\begin{aligned}f_e(v)=f_e(0)\exp\Biggl\{&-2\int_0^v\Biggl[\frac{dv'v'}{v_{te}^2}\Biggl\{\frac{3}{2}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{te}}\right)-\frac{v'}{v_{te}}\exp\left(-\frac{v'^2}{v_{te}^2}\right)\biggr]+\sum_i \frac{3\sqrt{\pi}}{4}\frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\biggl[\operatorname{erf}\left(\frac{v'}{v_{ti}}\right)\\&-\frac{2}{\sqrt{\pi}}\frac{v'}{v_{ti}}\exp\left(-\frac{v'^2}{v_{ti}^2}\right)\biggr]\Biggr\}\Biggl\{\frac{3}{2}\biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{te}}\right)-\frac{v'}{v_{te}}\exp\left(-\frac{v'^2}{v_{te}^2}\right)\biggr]\\&+\sum_i \frac{3\sqrt{\pi}}{4}\frac{Z_i^2 n_i v_{ti}^2}{n_e v_{te}^2}\biggl[\operatorname{erf}\left(\frac{v'}{v_{ti}}\right)-\frac{2}{\sqrt{\pi}}\frac{v'}{v_{ti}}\exp\left(-\frac{v'^2}{v_{ti}^2}\right)\biggr]\Biggr\}^{-1}\Biggr]\Biggr\}.\end{aligned}\tag{B3}$$

Note that by using the series expansion for $v'<v_{te}$ one finds

$$\begin{aligned}&\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{te}}\right)-\frac{v'}{v_{te}}\exp\left(-\frac{v'^2}{v_{te}^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{v'}{v_{te}}\right)^{2n+1}-\left(\frac{v'}{v_{te}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{v'}{v_{te}}\right)^{2n}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\left(\frac{2n}{2n+1}\right)\left(\frac{v'}{v_{te}}\right)^{2n+1}=\frac{2}{3}\left(\frac{v'}{v_{te}}\right)^3\\&\quad-\frac{2}{5}\left(\frac{v'}{v_{te}}\right)^5+\frac{1}{7}\left(\frac{v'}{v_{te}}\right)^7-O\left\{\left(\frac{v'}{v_{te}}\right)^9\right\}+\cdots.\end{aligned}\tag{B4}$$

By taking just the first term of this expansion, making the approximation $v\gg v_{ti}$, and defining the critical velocity v_c as before, the distribution function of Eq. (B3) reduces to the simpler form used in Sec. IV.

Even more accurate distribution functions could be found by using Eq. (B3) or a simplified form of it as the basis for further iterations with Eq. (B2).

Once a distribution function of the desired accuracy has been obtained, it can be used to find the effective electron temperature, as given by Eq. (B6), and the correction to the Spitzer ion–electron energy transfer rate,

$$\begin{aligned}\frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}}\approx&\frac{\sqrt{\pi}}{2}\frac{T_e}{(T_i-T_e)}\left(1+\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{3/2}\\&\times\int_0^\infty\Biggl\{dv\,v^2\frac{f_e(v)}{f_e(0)}\\&\times\biggl[\frac{2}{\sqrt{\pi}}\left(\frac{m_i}{m_e}\right)^{3/2}\sqrt{\frac{T_e}{T_i}}\exp\left(\frac{-v^2}{v_{ti}^2}\right)\\&-\frac{v_{te}}{v}\operatorname{erf}\left(\frac{v}{v_{ti}}\right)\biggr]\Biggr\}\left[\int_0^\infty dv\,v^2\frac{f_e(v)}{f_e(0)}\right]^{-1}.\end{aligned}\tag{B5}$$

Another possible improvement involves refining the definition of the electron temperature. For the case of significantly non-Maxwellian electrons, it is desirable to accompany the heat transfer expression by a definition of the effective electron temperature $\Theta_e \equiv 2\langle E_e \rangle/3$, where $\langle E_e \rangle$ is the mean energy per electron. One finds that

$$\begin{aligned}\Theta_e &\equiv \frac{2}{3}\frac{1}{n_e}\int_0^\infty (dv\;4\pi v^2)\left(\frac{1}{2}m_e v^2\right) f_e(v)\\ &= \frac{2}{3}T_e \frac{\int_0^\infty du\,u^4 f_e(u)}{\int_0^\infty du\,u^2 f_e(u)},\end{aligned} \tag{B6}$$

in which $u \equiv v/v_{te}$.

Numerical integration with Mathematica revealed that using the distribution function of Eq. (B3) produces only minute alterations in the graphs which were presented earlier. Likewise, plotting the heat transfer correction factor versus T_i/Θ_e (as opposed to T_i/T_e) only makes very slight alterations in the curves, since T_e and Θ_e only begin to diverge for large values of T_i/T_e, where the correction factor is nearly flat with respect to the temperature ratio.

More appreciable improvements might be gained from iterating the electron distribution function at least once more or by expressing all of the integrals in terms of Θ_e instead of T_e (being careful to maintain self-consistency with the new definition throughout the derivation), but these possibilities were not tested computationally, as the resulting expressions could not be numerically integrated within a reasonable time on the sort of computers presently available to the authors (Macintosh Quadra 610).

2. Improved answer for monoenergetic ion case

As in the case of Maxwellian ions, an even more accurate answer may be obtained by returning to the full nonlinear first-order differential equation for the electron distribution function, Eq. (49), and iterating. The next iteration expression for f_e written in terms of the previous iteration's less accurate expression, f_e^*, is

$$\begin{aligned} f_e(v) = f_e(0)\exp\Biggl\{ &-\int_0^v \Biggl[dv'v' \Biggl\{ 3\int_0^{v'} dv''\, f_e^*(v'')v''^2\\ &+\sum_i \frac{3Z_i^2 n_i}{4\pi}\frac{m_e}{m_i}\Theta(v'-v_i)\Biggr\}\\ &\times\Biggl\{ \int_0^{v'} dv''\, f_e^*(v'')v''^4 + v'^3\int_{v'}^\infty dv''\, f_e^*(v'')v''\\ &+\sum_i \frac{Z_i^2 n_i}{4\pi}\Biggl[v_i^2\Theta(v'-v_i)\\ &+\frac{v'^3}{v_i}\Theta(v_i-v')\Biggr]\Biggr\}^{-1}\Biggr]\Biggr\}. \end{aligned} \tag{B7}$$

As usual $f_e(0)$ is calculated directly from the normalization condition.

If one begins the iteration process by assuming that $f_e^*(v)$ is Maxwellian and characterized by the thermal velocity v_{te}, then the distribution function of Eq. (B7) becomes

$$\begin{aligned} f_e(v) = f_e(0)\exp\Biggl\{ &-2\int_0^v \Biggl[\frac{dv'v'}{v_{te}^2} \Biggl\{ \frac{3}{2}\Biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{te}}\right)\\ &-\frac{v'}{v_{te}}\exp\left(-\frac{v'^2}{v_{te}^2}\right)\Biggr] + \sum_i \frac{3\sqrt{\pi}}{4}\frac{Z_i^2 n_i}{n_e}\frac{m_e}{m_i}\\ &\times\Theta(v'-v_i)\Biggr\}\Biggl\{ \frac{3}{2}\Biggl[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{v'}{v_{te}}\right)\\ &-\frac{v'}{v_{te}}\exp\left(-\frac{v'^2}{v_{te}^2}\right)\Biggr] + \sum_i \frac{\sqrt{\pi}}{2}\frac{Z_i^2 n_i}{n_e}\\ &\times\Biggl[\frac{v_i^2}{v_{te}^2}\Theta(v'-v_i) + \frac{v'^3}{v_{te}^2 v_i}\Theta(v_i-v')\Biggr]\Biggr\}^{-1}\Biggr]\Biggr\}. \end{aligned} \tag{B8}$$

Note that by using the series expansion for $v_i < v' \ll v_{te}$ this distribution function reduces to the simpler one found given in the previous section.

Even more accurate distribution functions could be found by using Eq. (B8) or a simplified form of it as the basis for further iterations with Eq. (B7).

Once a distribution function of the desired accuracy has been obtained, it can be used to find the correction to the Spitzer ion–electron energy transfer rate,

$$\begin{aligned} \frac{P_{ie}}{(P_{ie})_{\text{Spitzer}}} \approx \sqrt{\frac{\pi}{2}}&\frac{T_e}{(\frac{2}{3}E_i - T_e)}\left(1+\frac{2}{3}\frac{m_e}{m_i}\frac{E_i}{T_e}\right)^{3/2}\\ &\times\left[\frac{m_i}{m_e}\frac{1}{v_i}\int_0^{v_i} dv\,v^2\frac{f_e(v)}{f_e(0)} - \int_{v_i}^\infty dv\,v\frac{f_e(v)}{f_e(0)}\right]\\ &\times\left[\sqrt{\frac{m_e}{T_e}}\int_0^\infty dv\,v^2\frac{f_e(v)}{f_e(0)}\right]^{-1}, \end{aligned} \tag{B9}$$

as well as the effective electron temperature, as given by Eq. (B6).

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[1] See AIP Document No. PAPS PHPAE-02-1873-8 for 8 pages of M. N. Rosenbluth, Energy Exchange Between Electrons and Ions. Gulf General Atomic Report GAMD-1710 (1960). Order by PAPS number and journal reference from American Institute of Physics, Physics Auxiliary Publication Service, Carolyn Gelbach, 500 Sunnyside Boulevard, Woodbury, New York 11797-2999. Fax: 516-576-2223, e-mail: janis@aip.org. The price is $1.50 for each microfiche (98 pages) or $5.00 for photocopies of up to 30 pages, and $0.15 for each additional page over 30 pages. Airmail additional. Make checks payable to the American Institute of Physics.

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