Relativistic Quantum Field Theory

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Any suggestions for improvements would be greatly appreciated.

Three quarks for Muster Mark!

–James Joyce, *Finnegans Wake (1939)*

I think I can safely say that nobody understands quantum mechanics.

–Richard Feynman, *The Character of Physical Law* (1965)

Overview

Relativistic quantum field theory (or field theory, for short) combines special relativity, which describes very fast things, and quantum mechanics, which describes very small things. The resulting theory correctly predicts the behavior of fundamental particles, which are small and often move at high speeds (or are in bound states with relativistic energies). A "toy theory" of spinless particles will be used to first introduce some of the basic techniques and results of field theory, since the spin of real particles makes calculations more complicated. Field theory will then be applied in succession to each of the four fundamental forces. Quantum electrodynamics is a field theory describing the electromagnetic force; it is relevant to phenomena such as Compton scattering and electron-positron annihilation. The field theory of the weak nuclear force describes phenomena such as the decay of neutrons (beta decay) and muons. Quantum chromodynamics describes the strong nuclear force and thus is relevant to the behavior of the quarks that compose particles like protons, neutrons, and pions. Finally, it will be shown that general relativity is equivalent to applying field theory to gravitation, although there remain obstacles to developing a complete quantum theory of gravity.

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1 Basic Principles: A Toy Theory

This section will introduce the basic principles of relativistic quantum field theory, including field equations, antiparticles, Feynman diagrams, and particle decays and reactions. To avoid complexities caused by the spins of real particles, a "toy theory" of spinless particles will be used here.

Before beginning, we should note that this entire summary uses cgs units; for methods of converting to mks units, see *Applied Mathematics* ?.?.

Familiarity with the index notation of special relativity as explained in *Relativity* ?.? will also be assumed. Briefly:

- Greek indices denote all four dimensions (space and time), whereas Latin indices denote spatial dimensions only. Repetition of the same index label on the same side of an equation denotes summation over all values of that index (the Einstein convention for implicit summation).
- The position vector of something in four-dimensional space-time is $x^\mu = (ct, x, y, z)$, where $ct \equiv x^0$.
- The reference metric of unwarped (Minkowski) space-time is:

$$\eta_{\mu\nu} = \eta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

- One raises and lowers indices by multiplying by $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. Examples: $X_\mu = \eta_{\mu\nu} X^\nu$; $X^\mu = \eta^{\mu\nu} X_\nu$.

1.1 Spin-0 Particle Field: Klein-Gordon Equation

1.1.1 Combining Quantum Physics and Special Relativity

In nonrelativistic quantum physics, particles act like waves and are represented by a wavefunction $\psi \sim e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}$. The wavefunction in relativistic quantum physics is essentially the same, except now it is called a **field** just to sound impressive.

Note that in the field or wavefunction, the energy E and time t variables are related to each other by getting multiplied together and divided by Planck's constant $\hbar$ in the exponential. The momentum $\mathbf{p}$ and position $\mathbf{x}$ variables are related to each other the same way. That's why you can take the Fourier transform of ψ as a function of t and get ψ as a function of E, or the Fourier transform of ψ as a function of $\mathbf{x}$ and get ψ as a function of $\mathbf{p}$. It's also why there is a Heisenberg uncertainty relationship between $\mathbf{x}$ and $\mathbf{p}$, and another uncertainty relationship between E and t. The product of the two variables is comparable to $\hbar$ in the exponential. If one variable becomes smaller (more accurately measured), the other variable must become larger (more uncertain) to keep the product constant. See *Nonrelativistic Quantum Physics* for more information.

In nonrelativistic quantum theory, ψ obeyed a wave equation called the Schrödinger equation which was derived from the relation between momentum and energy in nonrelativistic classical mechanics. Similarly, in relativistic quantum theory, ψ obeys a wave equation called the Klein-Gordon equation which is derived from the momentum-energy relation in special relativity:

Relativistic	**Nonrelativistic**
$\lvert\mathbf{p}\rvert^2 + m^2c^2 = \dfrac{E^2}{c^2}$	$\dfrac{\lvert\mathbf{p}\rvert^2}{2m} = E$
$\lvert\mathbf{p}\rvert^2 \psi + m^2c^2\psi = \dfrac{E^2}{c^2}\psi$	$\dfrac{\lvert\mathbf{p}\rvert^2}{2m}\psi = E\psi$
$\lvert\mathbf{p}\rvert^2 \psi \to -\hbar^2\nabla^2\psi \quad \& \quad E^2\psi \to -\hbar^2\dfrac{\partial^2}{\partial t^2}\psi$	$\lvert\mathbf{p}\rvert^2 \psi \to -\hbar^2\nabla^2\psi \quad \& \quad E\Psi \to i\hbar\dfrac{\partial}{\partial t}\psi$
$\left[\hbar^2\left(\dfrac{1}{c^2}\dfrac{\partial}{\partial t^2} - \nabla^2\right) + m^2c^2\right]\psi = 0$	$\left(-\dfrac{\hbar^2\nabla^2}{2m} - i\hbar\dfrac{\partial}{\partial t}\right)\psi = 0$
Klein-Gordon Eq.	Schrödinger Eq.

Note that potential energy was neglected in the above derivations for simplicity. Using the D'Alembertian operator $\Box^2 \equiv \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)$, the 4-dimensional space-time equivalent of the 3-dimensional Laplacian operator ∇^2, the Klein-Gordon equation is usually written as:

$$\left(\Box^2 + \frac{m^2c^2}{\hbar^2}\right)\psi = 0 \qquad \textbf{Klein}-\textbf{Gordon Eq}. \tag{2}$$

Combining relativity and quantum physics causes two interesting effects:

1. **Antiparticles.** The $\partial^2/\partial t^2$ in the Klein-Gordon Eq. (from the E^2 in the relativistic momentum-energy relation) means that a second solution $\psi \sim e^{i(\mathbf{p}\cdot\mathbf{x}+Et)/\hbar}$ is also acceptable. You can make this weird second solution look like the more conventional first one ($\psi \sim e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}$) if you pretend that either the energy is negative or time is negative (runs backwards), since E and t are multiplied. Unlimited negative energies are a physics nightmare, since you could extract energy from the universe indefinitely by simply creating more and more antimatter. The universe doesn't work that way, so one concludes that antiparticles have positive energy ($+E$) but act as if they are traveling backward in time ($-t$). An electron moving forward in time has a negative charge and thus travels toward a positive charge, but an electron moving backward in time would travel away from a positive charge–hence it looks to us who are moving forward in time as if it has a positive charge that is being repelled by the other positive charge. Ditto for magnetic fields and properties. Therefore, antiparticles have opposite magnetic and electric properties than their corresponding particles, but they still have positive energy. And since energy and mass are equivalent, their mass is positive too. Put differently, if antimatter had negative energy/mass, combining it with matter would yield zero net energy output. But we know that antimatter + matter makes a nice big boom, so their intrinsic masses/energies must be positive.

2. **Creation and annihilation of particles and antiparticles.** As shown in Fig. 1, a particle can make a U-turn in time to become an antiparticle (or vice versa), and to an observer steadily moving forward in time, the U-turns look like particle-antiparticle annihilation or creation events.

Another way to see that particles and antiparticles can be created and annihilated is to obtain the probability density for particles from the Klein-Gordon Eq., using the same approach that yielded the probability density $\rho = \psi^*\psi$ from the Schrödinger Eq. in nonrelativistic quantum theory. The first step is to multiply Eq. (2) by ψ^* and take the complex conjugate of the resulting equation:

$$\psi^*(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\psi = 0 \tag{3}$$
$$\psi(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\psi^* = 0\ , \tag{4}$$

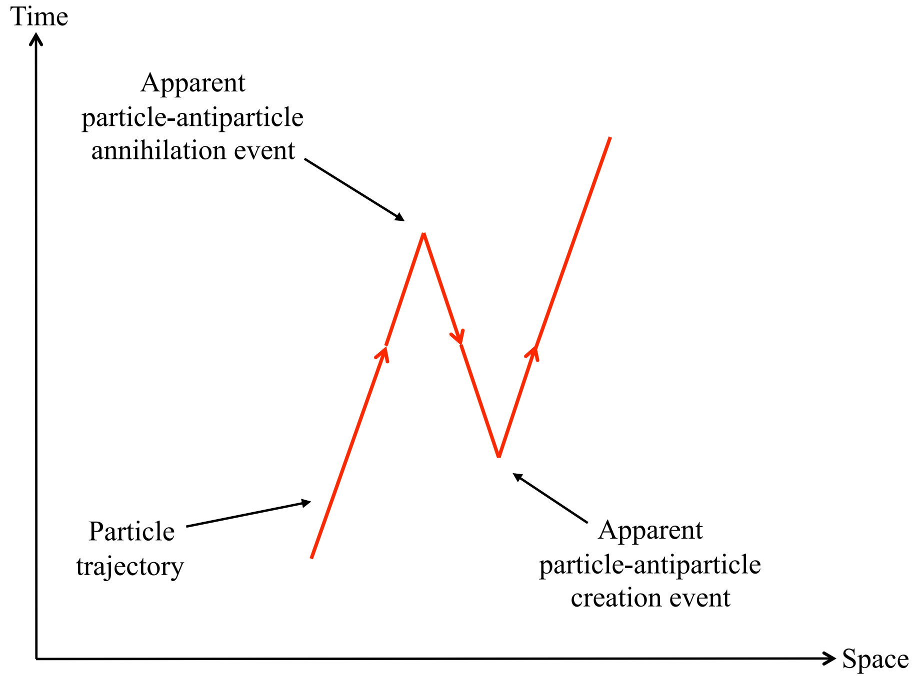

Figure 1. Creation and annihilation of particles and antiparticles. A particle traveling forward in time can make a "U-turn" in time to become an antiparticle traveling backward in time, and vice versa. To an observer steadily moving forward in time, these U-turns look like the creation or annihilation of particle-antiparticle pairs.

where $\partial_\mu \partial^\mu \equiv \Box^2$. Then subtract Eq. (4) from Eq. (3):

$$\begin{aligned} \psi^* \hbar \partial_\mu \hbar \partial^\mu \psi - \psi \hbar \partial_\mu \hbar \partial^\mu \psi^* &= 0 \ , \quad \text{or} \\ \partial_\mu (\psi^* \hbar \partial^\mu \psi - \psi \hbar \partial^\mu \psi^*) &= 0 \ , \end{aligned} \tag{5}$$

since $\partial_\mu \psi^* \partial^\mu \psi - \partial_\mu \psi \partial^\mu \psi^* = 0$. Equation (5) is in the form of a conservation equation, $\partial_\mu j^\mu = 0$, where the conserved current $j^\mu = \psi^* \hbar \partial^\mu \psi - \psi \hbar \partial^\mu \psi^*$. Thus the probability density is

$$\rho = j^0 = \frac{i\hbar}{c} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \ , \tag{6}$$

where the i has been included to make the expression real. Equation (6) gives $\rho = 2E/c$ for $\psi = e^{i(\mathbf{p}\cdot\mathbf{x} - Et)/\hbar}$, so a field normalized to one particle per unit volume is

$$\psi \ = \ \sqrt{\frac{c}{2E}} \, e^{i(\mathbf{p}\cdot\mathbf{x} - Et)/\hbar} \ . \tag{7}$$

Antiparticles have $\psi \sim e^{i(\mathbf{p}\cdot\mathbf{x} + Et)/\hbar}$, so Eq. (6) assigns them negative probability densities. In other words, the conserved quantity in Eq. (6) is the *net* probability density of particles *minus* antiparticles. A particle and an antiparticle can be freely created together or annihilated together without affecting this conservation law. As a result, quantum fields do not represent a defined number of particles, just as the electromagnetic field does not represent a fixed number of photons.

1.1.2 Virtual Particles and Fundamental Forces

The Klein-Gordon equation in Eq. (2) is for *free particles*, since the derivation neglected potential energy from interactions with other particles or fields. Interactions would add a "source" term on the right side of Eq. (2). A simple example is a point source of strength g located at $\mathbf{r} = 0$:

$$\left(\frac{1}{c^2} \frac{\partial}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2} \right) \psi = g\delta(\mathbf{r}) \tag{8}$$

For static fields ($\partial^2/\partial t^2 = 0$), this equation may be rewritten and solved:

$$\begin{aligned} \nabla^2 \psi &= \frac{m^2c^2}{\hbar^2} \psi - g\delta(\mathbf{r}) \quad , \quad \text{or} \\ \psi(\mathbf{r}) &= \frac{g}{4\pi r} \exp\left(- \frac{mc}{\hbar}\, r \right) . \qquad\qquad \textbf{Yukawa potential} \end{aligned} \tag{9}$$

The physical interpretation of this result is that the source $\delta(\mathbf{r})$ is surrounded by particles of the field ψ. The ψ particles have zero energy ($\partial^2\psi/\partial t^2 = 0$), whereas they would need an added energy $\Delta E = \sqrt{(mc^2)^2 + |\mathbf{p}|^2 c^2}$ to really exist. Thus they are not **real** particles which can exist indefinitely, but rather **virtual** particles which appear and disappear within a time Δt governed by the uncertainty relation $(\Delta t)(\Delta E) \sim \hbar$. If there were another source nearby, these virtual ψ particles would interact with that source to cause an attractive or repulsive force between the two sources. For example, the electromagnetic force is caused by virtual photons being exchanged by two objects, gravity is caused by virtual gravitons being exchanged, the strong nuclear force is caused by virtual gluons or pions, and the weak nuclear force is caused by virtual W or Z particles.

Therefore, the field in Eq. (9) may be interpreted as an attractive or repulsive potential surrounding the source $\delta(\mathbf{r})$. It falls off exponentially with distance in a manner depending on the mass of the virtual particles, since more massive virtual particles are further from having enough energy to really exist and hence cannot travel very far before they vanish. Yukawa first used this type of potential to describe the strong nuclear force caused by massive pions. For massless particles such as photons, Eq. (9) reduces to the form of the Coulomb potential from electrostatics.

In general, virtual particles can have any four-momentum p^μ, whereas real particles must have a four-momentum that satisfies the relativistic momentum-energy relation

$$\begin{aligned} p^2 &\equiv p^\mu p_\mu = \frac{E^2}{c^2} - |\mathbf{p}|^2 \\ &= m^2c^2 . \end{aligned} \tag{10}$$

Because Eq. (10) describes a spherical shell of radius mc in Minkowski four-momentum space, real particles are often said to be "on their mass shell" and virtual particles are "off-shell."

For reasons known only to nature (**or is there an explanation?**), the fundamental particles such as electrons and quarks that make up matter all have half-integer spin and thus are fermions. That choice of nature means that the particles such as photons and gravitons which mediate forces between particles of matter must have integer spin and be bosons. If a fermion particle of matter emitted or absorbed a force-mediating virtual particle that was a fermion, the matter particle would lose or gain a half-integer of spin and hence change from a fermion to a boson. Only by emitting and absorbing bosons can the fermions remain fermions.

1.1.3 Field Formalities

Uncharged particles may be represented by a real field $\phi \sim \mathrm{Re}\{e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}\} = \mathrm{Re}\{e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{-iEt/\hbar}\}$. Barring interactions, such a particle has a conserved momentum $\mathbf{p}$ (connected with invariance of the field under spatial translations $\mathbf{\Delta x}$, as shown in the classical mechanics summary) and energy E (connected with invariance under time translations Δt). Note how each conserved quantity ($\mathbf{p}$ or E) is mathematically related to its corresponding variable of translation ($\mathbf{x}$ or t) in the field by a factor like exp[i(conserved quantity)(variable representing a dimension)], bringing us up to a total of four dimensions (three for space and one for time).

Charged particles have an additional conserved quantity: their charge q. This can be represented by modifying the field to be $\psi \sim e^{iq\theta}\mathrm{Re}\{e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}\}$, where θ is a phase variable just as $\mathbf{x}$ and t are variables. Essentially we are invoking a new fifth dimension θ to pair with the conserved charge q. While this notion may sound strange, it can be traced back to the earliest Kaluza-Klein and Einstein unified field theories. If there is a fifth dimension θ, why can't we move in it, just as we can move in space and time? Although momentum and energy are each conserved overall, a specific particle can change its own momentum and energy by swapping some with other particles as the particles move in space and time. In contrast, electrons cannot change their charge, and neither can quarks. Perhaps the reality of a fifth θ dimension makes charge exist, but the fact that we can't freely move in the theta dimension keeps anybody from changing their charge. Or as Einstein and others speculated, maybe the fifth dimension just goes in a circle (you keep coming back to where you started) and the circle is so small (order of $\hbar$) that we can't even detect that we travel around it. Speculations aside, the bottom line is that charged particles are represented by a complex field. Usually for a charged field, one simply defines $\psi \sim e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}$ to be complex instead of explicitly including q in it. This sweeps any extra dimension θ under the rug.

A more formal view of the need for complex fields uses Eq. (6), the conserved net density of particles minus antiparticles. This density is automatically zero for real (uncharged) fields, so uncharged particles are their own antiparticles (just as the negative of the number 0 is also 0), and the number of particles in a real field need not be conserved. If particles are charged, there must exist antiparticles with the opposite charge, and one should be able to have more particles than antiparticles, or vice versa, if one wished. The conserved density of Eq. (6) can be nonzero for complex (charged) fields, and when multiplied by the particle charge it may be viewed as the *charge density* since it counts particles minus antiparticles.

While particles without spin can be represented by scalar fields like those above, particles with spin must be represented by vector fields (for spin-$\frac{1}{2}$ particles like electrons and spin-1 particles like photons) or tensor fields (for spin-2 particles like gravitons) to indicate the spin direction. This will be explained more in Sections 2-5.

Using techniques from the nonrelativistic quantum summary, ϕ and ψ may be second-quantized:

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{c}{2E_\mathbf{p}}}\left(\hat{a}_\mathbf{p}e^{-ip\cdot x} + \hat{a}^\dagger_\mathbf{p}e^{+ip\cdot x}\right) \tag{11}$$

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{c}{2E_\mathbf{p}}}\left(\hat{b}_\mathbf{p}e^{-ip\cdot x} + \hat{c}^\dagger_\mathbf{p}e^{+ip\cdot x}\right) \qquad \psi^*(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{c}{2E_\mathbf{p}}}\left(\hat{c}_\mathbf{p}e^{-ip\cdot x} + \hat{b}^\dagger_\mathbf{p}e^{+ip\cdot x}\right)$$

The real field ϕ has the operator $\hat{a}^\dagger_\mathbf{p}$ that creates ϕ particles with momentum $\mathbf{p}$ and energy $E_\mathbf{p}$ and the operator $\hat{a}_\mathbf{p}$ that annihilates them. For the complex field ψ, $\hat{b}^\dagger_\mathbf{p}$ and $\hat{b}_\mathbf{p}$ create and annihilate charged particles, and $\hat{c}^\dagger_\mathbf{p}$ and $\hat{c}_\mathbf{p}$ create and annihilate antiparticles with the opposite charge.

It is customary to construct a Lagrangian that when inserted into the Euler-Lagrange equations (see *Classical Mechanics* ?.?) gives the appropriate quantum field equation. The Lagrangian L is just a system's kinetic energy *minus* its potential energy. For example, the Klein-Gordon Eq. (2) for a complex field may be obtained from the Lagrangian density $\mathcal{L}$ (Lagrangian per volume):

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\psi)^*(\partial^\mu\psi)}_{\text{Kinetic energy}} - \underbrace{\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\psi^*\psi}_{\text{Potential energy stored as particle mass}} \quad . \tag{12}$$

Integrating the Lagrangian density over space gives the Lagrangian, $L = \int d^3x\mathcal{L}$. Why does relativistic quantum physics focus more on the Lagrangian $L =$ (kinetic energy) - (potential energy), whereas nonrelativistic quantum physics focuses more on the Hamiltonian or total energy, $L =$ (kinetic energy) + (potential energy)? Mostly that is just a difference in custom–you could take other paths to arrive at the same final answer. What physical justification there is would go like this: When $\mathcal{L}$ is integrated over all space and time (that is called the action) and one tries to make the result as small as possible (principle of least action), the laws of motion governing that system pop out–Newton's laws for particles, Maxwell's equations for photons, etc. In other words, averaged over space and time, the universe tries to make the kinetic energy of a system as small as possible and the potential energy as large as possible, as if things are naturally seeking a steady place to rest. (Waving hands wildly...)

As a detailed example, the rest of Section 1 will consider a "toy theory" in which charged "nucleons" denoted by the complex field ψ interact with uncharged "pions" denoted by the real field ϕ. The complete Lagrangian density for this theory is

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\psi)^*(\partial^\mu\psi) - \frac{1}{2}\left(\frac{m_n c}{\hbar}\right)^2\psi^*\psi}_{\substack{\text{Lagrangian of free charged}\\ \text{“nucleons” with mass } m_n}} + \underbrace{\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\left(\frac{m_\pi c}{\hbar}\right)^2\phi^2}_{\substack{\text{Lagrangian of free uncharged}\\ \text{“pions” with mass } m_\pi}} - \underbrace{g\phi\psi^*\psi}_{\substack{\text{Nucleon-pion}\\ \text{interaction energy}}} \tag{13}$$

The interaction term of the Lagrangian density, $\mathcal{L}_{\text{Int}} = -g\phi\psi^*\psi$, is *minus* the potential energy due to nucleon-pion interactions. Taking the complex conjugate of one of the ψ fields ensures the interaction energy will be real. The **coupling constant** g governs how strongly the fields interact.

1.2 Feynman Diagrams

The processes of particles reacting with each other or decaying into other particles may be represented by schematic illustrations known as **Feynman diagrams**. Mathematical rules will now be derived in order to calculate reaction cross sections and decay rates from Feynman diagrams.

1.2.1 Scattering Matrix

To derive Feynman diagrams and rules, one can begin by considering particle interactions in general. The interaction Lagrangian density is $\mathcal{L}_{\text{Int}} = -E_{\text{Int}}/\text{volume}$, where E_{Int} is the interaction energy among the particles. For example, the interaction Lagrangian for the toy theory is

$$\mathcal{L}_{\text{Int}} = -g\phi\psi^*\psi \ . \tag{14}$$

The physical meaning of this interaction Lagrangian is that each field in Eq. (14) represents a particle meeting in the same interaction "vertex." ψ could be a nucleon coming into the vertex or an antinucleon leaving it. Similarly, ψ^* could be either a nucleon leaving or an antinucleon coming into the interaction vertex.

Either one ϕ pion comes into the interaction vertex or one pion leaves it (but not both). Why don't we say that ϕ could represent a particle going one way or antiparticle going the other way, just like ψ^* and ψ^*? As mentioned in Section 1.1.3, uncharged particles (real fields) are their own antiparticles. For example, you never hear about anti-photons or anti-gravitons, because there aren't any. To use a math analogy, numbers like 1, 2, 3 have "anti" version -1, -2, -3. 0 doesn't have an anti version, or more to the point the anti-version of 0 is just 0. The equivalent of photons, gravitons, etc. in this toy theory is the pion. Thus the ϕ in Eq. (14) could represent either a pion coming to that interaction vertex or a pion leaving it.

This same basic technique works for any kind/number of particle interactions, so it is very handy. You multiply by the wave function for each particle, because when you square the resulting amplitude to find the probability of a particular reaction occurring, each wave function in effect gets squared. Thus the probability of the whole event occurring is proportional to the probabilities that all the necessary initial, intermediate, and final particles are really there (their wave functions/fields squared). In real physics theories (Sections 2-5), each vertex also picks up a tensor or matrix to show how the three-dimensional spatial components of each field interact with those of the other fields, making the calculations much scarier, but the basic principle is the same.

Suppose that the initial state of a system of particles is $|\Psi_{\text{initial}}\rangle$ (say at $t = -\infty$, just to be as initial as possible) and the final state is $\langle\Psi_{\text{final}}|$ (at $t = +\infty$). The time evolution from the initial to the final states due solely to interactions among the particles (interaction picture) is given by the **scattering matrix** S (using $x_0 \equiv ct$ in the integral):

$$\begin{aligned} S \equiv \langle\Psi_{\text{final}}|\, S\, |\Psi_{\text{initial}}\rangle &\sim \exp\left(-\frac{iE_{\text{Int}}t}{\hbar}\right) \\ &= \exp\left(\frac{i}{\hbar c}\int d^4x\, \mathcal{L}_{\text{Int}}\right) \end{aligned} \tag{15}$$

The scattering matrix S tells how the fields change over time just because of their interactions, which are described by the potential energy term $E_int = -L_int$ from Eq. (14). In nonrelativistic quantum physics and now in relativistic quantum physics, time variation due to energy always goes like $e^{-iEt/\hbar}$, so we plug in E_int here. To go from the first to the second line of Eq. (15), note that the sign flips in going from E_int to L_int, we integrate over all three spatial dimensions d^3x since here L_int is a density per volume, and $t \Longrightarrow \int dt \Longrightarrow \int dx_0/c$ (calling the time dimension x_0 and measuring it in meters to make it look mathematically more like the spatial dimensions). Thus d^4x means we are now integrating over the three spatial dimensions plus the time dimension.

Using the expansion $e^x = 1 + \sum_{n=1}^{\infty}(x^n/n!)$, the scattering matrix may be rewritten as:

$$S = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar c}\int d^4x\, \mathcal{L}_{\text{Int}}\right)^n \tag{16}$$

To be truly inclusive, the sum must be done not only over all n, but also over all contractions of the fields (identification of two fields in a product as representing the same particle at different times) and all permutations of which interaction vertex is which. Since there are $n!$ permutations of n vertices, the sum over permuations cancels the $1/n!$, leaving only:

$$S = 1 + \sum_{n=1}^{\infty}\sum_{\text{contractions}}\left(\frac{i}{\hbar c}\int d^4x\, \mathcal{L}_{\text{Int}}\right)^n \tag{17}$$

Using $\mathcal{L}_{\text{Int}}$ for the toy theory and using overhead lines to connect contracted fields (two different fields that represent the same particle moving between two different vertices, as will be explained in more detail in Section 1.2.2), one obtains

$$
\begin{aligned}
S \; &= \; 1 + \sum_{n=1}^{\infty} \sum_{\text{contractions}} \left(-\frac{ig}{\hbar c} \int d^4x \; \phi \psi^* \psi \right)^n \\
&= \; 1 && \textbf{(a)} \\
&\quad -\frac{ig}{\hbar c} \int d^4x \; \phi \psi^* \psi && \textbf{(b)} \\
&\quad + \left(-\frac{ig}{\hbar c} \right)^2 \int d^4x_1 \; \overline{\phi\,(x_1) \psi^*(x_1) \psi(x_1) \int d^4x_2 \; \phi}(x_2) \psi^*(x_2) \psi(x_2) && \textbf{(c)} \\
&\quad + \left(-\frac{ig}{\hbar c} \right)^2 \int d^4x_1 \; \phi(x_1) \overline{\psi^*\,(x_1) \psi(x_1) \int d^4x_2 \; \phi(x_2) \psi^*(x_2) \; \psi}(x_2) && \textbf{(d)} \\
&\quad + \left(-\frac{ig}{\hbar c} \right)^2 \int d^4x_1 \; \phi(x_1) \psi^*(x_1) \overline{\psi\,(x_1) \int d^4x_2 \; \phi(x_2) \; \psi^*}(x_2) \psi(x_2) && \textbf{(e)} \\
&\quad + \ldots && (18)
\end{aligned}
$$

The physical meaning of each term in Eq. (18) is shown in Fig. 2.

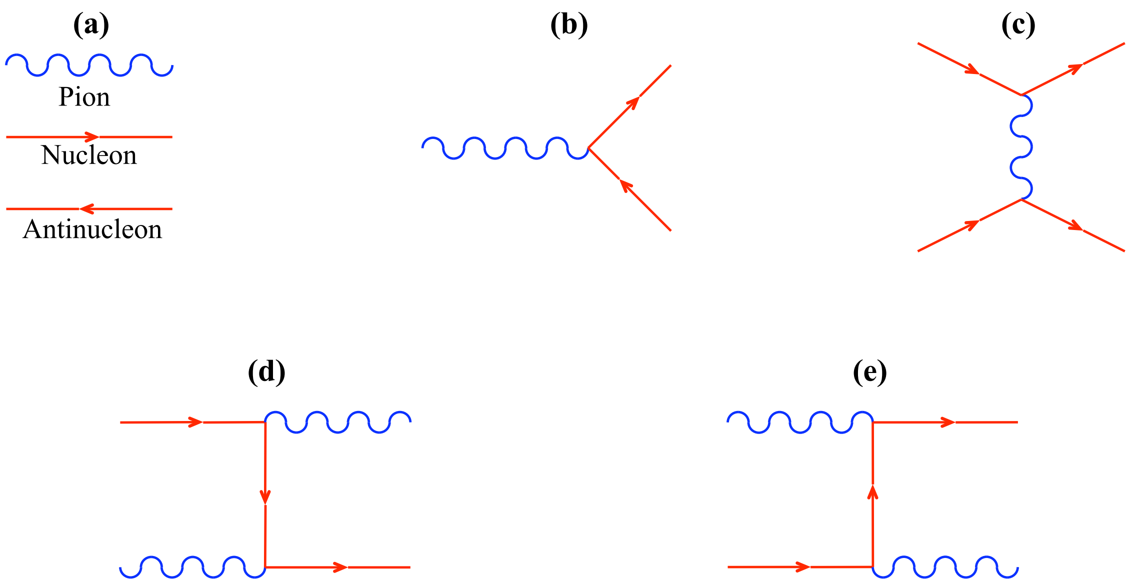

Figure 2. Physical meaning of the terms in Eq. (18). These Feynman diagrams show that each term represents possible interactions among pions (wavy lines), nucleons (lines with arrows pointed one way), and antinucleons (lines with arrows pointed the other way). The first term, 1, represents the possibility that the particles do not interact. The second term represents one interaction vertex among the particles. Higher terms are interactions with two or more vertices. Time flows in one direction in each diagram, but that direction depends on the process being described. For example, the one-vertex term represents a pion turning into a nucleon and an antinucleon if time flows to the right, a nucleon-antinucleon pair turning into a pion if time flows to the left, a nucleon absorbing or emitting a pion if time flows upward, or an antinucleon absorbing or emitting a pion if time flows downward.

Thus the scattering matrix S is a complex exponential of time [Eq. (15)], but that exponential can be rewritten as a Taylor series [Eq. (17)]. Physically, the terms of the Taylor series represent all possible interactions, including different numbers of vertices (starting with the fewest) and all possible contractions of fields. Figure 2 illustrates the first few terms; after those would come all possible interactions with 3 vertices, then 4, then 5, etc. Terms with fewer vertices are more important than those with more, since each vertex brings a multiplicative factor $g \ll 1$ (except in quantum gravity–see Section 5). Basically, the series solution for S contains everything that could possibly happen, given any number and type of initial and final particles. In practice, you know what initial and final particles you have (e.g., you want to calculate the repulsion between two nucleons, so you know there are two initial incoming nucleons, two final outgoing nucleons, and only virtual mesons), so you can ignore all but one or a few terms in the series. What you end up with is just a number, a complex amplitude. Multiply that by its complex conjugate and you get the probability that interaction will actually occur (within a multiplicative factor–see Section 1.3).

1.2.2 Evaluation of One- and Two-Vertex Terms

For now, consider only the one-vertex term in Eq. (18), which can represent a pion turning into a nucleon and an antinucleon. Using the momenta shown in Fig. 3, each particle's field may be written as

$$\phi(x) = e^{-ip_1\cdot x/\hbar} \qquad \psi^*(x) = e^{ip_2\cdot x/\hbar} \qquad \psi(x) = e^{ip_3\cdot x/\hbar} \tag{19}$$

Note that the four-momentum of the outgoing particles is explicitly written as the opposite sign.

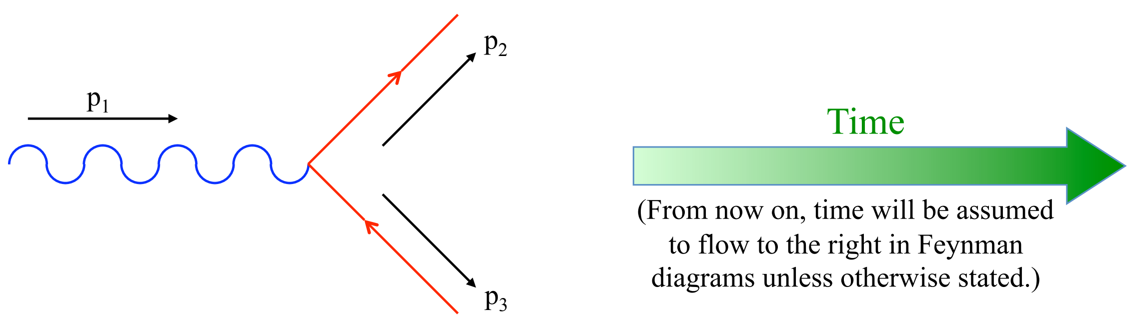

Figure 3. Pion decay. In this toy theory, a pion with four-momentum p_1 can decay into a nucleon with momentum p_2 and an antinucleon with momentum p_3 if $m_\pi > 2m_n$.

Using these expressions for the fields, this 1-vertex term in S may be evaluated:

$$\begin{aligned} S_{1-vertex} &= -\frac{ig}{\hbar c}\int d^4x\ \phi(x)\psi^*(x)\psi(x) \ = \ -\frac{ig}{\hbar c}\int d^4x\ e^{-i(p_1-p_2-p_3)\cdot x/\hbar} \\ &= -\frac{ig\hbar^3}{c}\,(2\pi)^4\ \delta^4(p_1-p_2-p_3)\ , \end{aligned} \tag{20}$$

where the properties of the delta function were used:

$$\int d^4x\ e^{-ip\cdot x/\hbar} = (2\pi)^4\delta^4(p/\hbar) = \hbar^4(2\pi)^4\delta^4(p)\ . \tag{21}$$

A brief explanation of Eq. (21) is in order. If the imaginary exponent of e is nonzero, the $e^{i \text{ whatever}}$ oscillates in space and time, just as a good old-fashioned wave function is supposed to do. Integrating the oscillations over space and time gives an end result of zero–half the time the oscillation is up, half the time it's down, but when averaged over everything it is zero. The only way to get a nonzero final answer is if the exponent of e is zero ($e^0 = 1$). That is the definition of a delta function–it is zero everywhere except at one precise value of its argument. If the oscillations are integrated over all x, one obtains a delta function of p. If the oscillations are integrated over all p instead, one gets a delta function of x, as will be used in Eqs. (24-25).

Next consider the two-vertex term shown in Fig. 4, which represents nucleon-nucleon scattering:

$$\begin{aligned} S_{nn} &= \left(-\frac{ig}{\hbar c}\right)^2 \int d^4x_1\, \phi\,(x_1)\psi^*(x_1)\psi(x_1) \int d^4x_2\ \phi(x_2)\psi^*(x_2)\psi(x_2) \\ &= \left(-\frac{ig}{\hbar c}\right)^2 \int d^4x_1 \int d^4x_2\, \phi\,(x_1)\ \phi(x_2)\psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) \end{aligned} \tag{22}$$

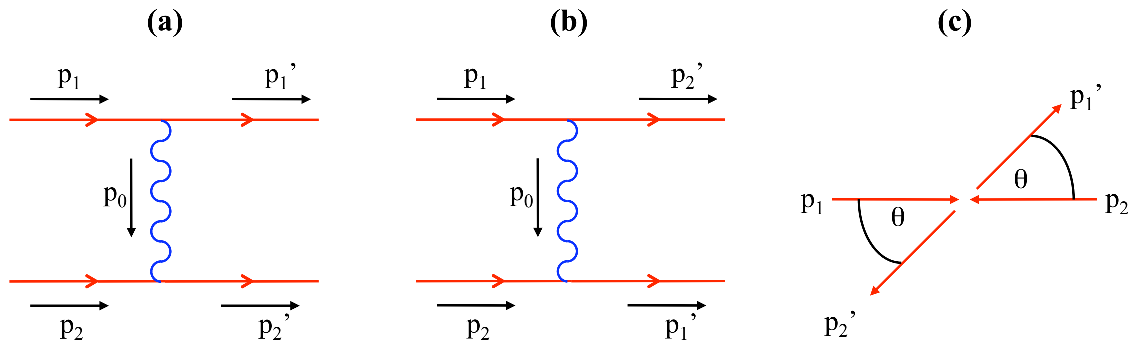

Figure 4. Nucleon-nucleon scattering. Feynman diagrams **(a)** and **(b)** both contribute to the scattering. The only difference between them is that the nucleons can "swap identities." **(c)** The relative angles of the nucleons' incoming and outgoing three-momenta are shown.

The contraction between the fields $\phi(x_1)$ and $\phi(x_2)$ means those fields represent the same particle, or in other words a ϕ particle created at point x_1 in space-time in the first interaction vertex travels to and is absorbed at the second interaction x_2 (or vice versa). If $\phi(x_1)$ and $\phi(x_2)$ are not contracted, they refer to different ϕ particles and each is under a separate, independent integral. But if they are contracted, $\phi(x_2)$ depends on how far away the field was emitted at x_1. Thus the two integrals are interrelated and the two integral signs are pulled to the front to apply to the whole expression in the second line of Eq. (22). ϕ represents an uncharged mediator particle (like a photon), so it is a real field (Section 1.1.3) and is the same as its complex conjugate. You do get into complex numbers when contracting charged particle fields; in fact Section 2 shows how messy that gets for realistic electrons, as opposed to the pretend particles of Section 1. In the end, though, all of this is just to calculate an amplitude, and then that amplitude is multiplied by its complex conjugate to get a real probability that the reaction will occur.

In general, for the creation of a ϕ field at x_1, the field at other points is $\phi(x) \equiv G(x - x_1)$, where $G(x - x_1)$ is the field solution of the Klein-Gordon equation with a unit point source at x_1:

$$[\hbar^2 \Box^2 + m^2c^2]\, G(x - x_1) \ = \ \delta^4(x - x_1)\,. \tag{23}$$

Math nerds may recognize G to be a Green's function; other folks shouldn't worry about that. Anyway, G may be found by Fourier-transforming Eq. (23) to momentum-space:

$$[-p^2 + m^2c^2]\,\tilde{G} \;=\; e^{-ip\cdot x_1/\hbar} \qquad\Longrightarrow\qquad \tilde{G} \;=\; -\frac{e^{-ip\cdot x_1/\hbar}}{p^2 - m^2c^2}\;, \tag{24}$$

where $\tilde{G}$ is the Fourier transform of G. Transforming back to regular space yields

$$G(x - x_1) \;=\; \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x/\hbar}\,\tilde{G} \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2c^2}\, e^{-ip\cdot(x_1 - x)/\hbar}\;. \tag{25}$$

Thus the field at x_2 caused by the creation of ϕ at x_1 (with the physical meaning of each factor) is

$$\overline{\phi\;(x_1)\;\phi}(x_2) \;=\; G(x_2 - x_1) \;=\; \underbrace{\int \frac{d^4p_0}{(2\pi)^4}}_{\substack{\text{Integrate over}\\ \text{all possible}\\ \text{momenta}\\ \text{(particle is virtual)}}} \quad \underbrace{\frac{i}{p_0^2 - m_\pi^2c^2}}_{\substack{\text{Smaller if far}\\ \text{off mass shell}\\ \text{(less likely}\\ \text{to exist)}}} \quad \underbrace{e^{-ip_0\cdot(x_1 - x_2)/\hbar}}_{\substack{\text{Phase factor}\\ \text{from wave}\\ e^{ip_0\cdot x/\hbar}\text{ going}\\ \text{from } x_1 \text{ to } x_2}}\;. \tag{26}$$

Since the contraction will be used to calculate an amplitude, constant phase factors do not matter, so the expression has been multiplied by $(-i)$ to agree with convention.

The weighting factor $i/(p^2 - m^2c^2)$ in Eq. (26) is called the propagator, since it determines how likely a virtual particle is to propagate from one interaction vertex to the other. The propagator has turned out to be directly related to the momentum-space wave equation for the free field:

$$\begin{aligned} [f(p)]\phi \;=\; 0 \qquad & \text{momentum-space wave eq. for free field} \\ & \Longrightarrow \quad \text{propagator} \;=\; -i \;\times\; \text{inverse of } [f(p)] \end{aligned} \tag{27}$$

For this toy theory $[f(p)] = [-p^2 + m^2c^2]$, but the general relation in Eq. (27) will prove useful in later sections for developing propagators for virtual particles in real field theories.

The other fields are

$$\begin{aligned} \psi(x_1) &= e^{-ip_1\cdot x_1/\hbar} & \qquad \psi^*(x_1) &= e^{ip_1'\cdot x_1/\hbar} \\ \psi(x_2) &= e^{-ip_2\cdot x_2/\hbar} & \qquad \psi^*(x_2) &= e^{ip_2'\cdot x_2/\hbar} \end{aligned}$$

Substituting these expressions for the fields into Eq. (22), one finds

$$\begin{aligned} S_{nn} \;&=\; \left(-\frac{ig}{\hbar c}\right)^2 \int d^4x_1 \int d^4x_2 \int \frac{d^4p_0}{(2\pi)^4}\,\frac{i}{p_0^2 - m_\pi^2c^2}\; e^{-i[p_0\cdot(x_1 - x_2) - p_1'\cdot x_1 + p_1\cdot x_1 - p_2'\cdot x_2 + p_2\cdot x_2]/\hbar} \\ &=\; \left(-\frac{ig}{\hbar c}\right)^2 \int d^4x_1 \int d^4x_2 \int \frac{d^4p_0}{(2\pi)^4}\,\frac{i}{p_0^2 - m_\pi^2c^2}\; e^{-i(p_1 + p_0 - p_1')\cdot x_1/\hbar} e^{-i(p_2 - p_0 - p_2')\cdot x_2/\hbar} \\ &=\; \left(-\frac{ig}{\hbar c}\right)^2 \int \frac{d^4p_0}{(2\pi)^4}\,\frac{i}{p_0^2 - m_\pi^2c^2}\; \hbar^4(2\pi)^4\delta^4(p_1 + p_0 - p_1')\hbar^4(2\pi)^4\delta^4(p_2 - p_0 - p_2')\;. \end{aligned} \tag{28}$$

External observers only see the overall result of this interaction instead of the individual vertices. Thus the result of Eq. (28) may be put into a form resembling the single interaction of Eq. (20):

$$S_{nn} = -\frac{iA\hbar^3}{c}\,(2\pi)^4\,\delta^4(p_1 + p_2 - p_1' - p_2')\;. \tag{29}$$

Here g has been replaced by an interaction amplitude A, which can be computed from Eq. (28).

1.2.3 Feynman Rules

Generalizing from the calculations in the preceding two sections, one can find the amplitude of any process in this toy theory by following these **Feynman rules**:

1. Label the four-momentum p_i of each line, using an arrow to denote the positive direction.

2. For each virtual particle (internal line), integrate over all possible four-momenta with a weighting factor (propagator) that makes momenta further from the particle's mass shell less likely to occur:

$$\int \frac{d^4 p_i}{(2\pi)^4} \frac{i}{p_i^2 - m_i^2 c^2} ,$$

where p_i and m_i are the four-momentum and mass of that virtual particle.

3. For each interaction vertex, include a factor

$$(-ig)(2\pi)^4 \delta^4(\sum p_{\text{in}} - \sum p_{\text{out}}) ,$$

in which p_{in} and p_{out} are the momenta entering and leaving the vertex, respectively. The g accounts for the interaction strength and the δ^4 function ensures that energy and momentum are conserved.

4. The result is

$$-iA(2\pi)^4 \delta^4(\sum p_{\text{in}} - \sum p_{\text{out}}) ,$$

where p_{in} and p_{out} are the initial and final momenta.

The probability of the process happening will be proportional to $|A|^2$, just as wavefunctions are squared to get probabilities in nonrelativistic quantum physics.

1.3 Examples

1.3.1 General Formula for Decay Rates

Consider the decay of particle 1 into multiple particles, $1 \to 2+3+...$ The decay rate (probability of decay per time) is $\Gamma \propto |A|^2$:

$$\Gamma \;=\; S\;\frac{c}{\hbar}\;\underbrace{\left[\int \frac{d^3\mathbf{p_2}}{(2\pi)^3}\int \frac{d^3\mathbf{p_3}}{(2\pi)^3}...\right]}_{\substack{\text{Integrate over phase}\\ \text{space of final momenta}}}\;\underbrace{(2\pi)^4\delta^4(p_1-p_2-p_3-...)}_{\substack{\text{Conserve energy}\\ \text{\& momentum}}}\;\underbrace{\left(\frac{c}{2E_1}\frac{c}{2E_2}\frac{c}{2E_3}...\right)}_{\substack{\text{Normalize}\\ \text{fields}}}\;|A|^2\;, \tag{30}$$

in which the degeneracy factor S is a product of $1/N!$ for each N identical final particles.

The decay rate may be simplified for two-body decay in which a stationary particle of mass m_1 decays into two particles of masses m_2 and m_3 with $m_1 > m_2 + m_3$. Since the initial particle is motionless, $\mathbf{p_1} = 0$ and $E_1 = m_1c^2$. It is also helpful to rewrite the delta function in the form $\delta^4(p_1 - p_2 - p_3) = c\delta(E_1 - E_2 - E_3)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3})$:

$$\begin{aligned}\Gamma &= \frac{Sc^3}{32\pi^2\hbar m_1}\int \frac{d^3\mathbf{p_2}d^3\mathbf{p_3}}{E_2E_3}\delta(m_1c^2-E_2-E_3)\delta^3(-\mathbf{p_2}-\mathbf{p_3})\,|A|^2\\ &= \frac{Sc^3}{32\pi^2\hbar m_1}\int \frac{d^3\mathbf{p_2}}{E_2E_3}\delta(m_1c^2-E_2-E_3)\,|A|^2\;,\end{aligned} \tag{31}$$

where integrating $d^3\mathbf{p_3}$ over the δ^3 function gave $\mathbf{p_3} = -\mathbf{p_2}$.

Since the remaining delta function is in terms of energy, $d^3\mathbf{p_2}$ will be rewritten in terms of energy. First change to spherically symmetric coordinates in momentum space,

$$d^3\mathbf{p_2} = d\Omega\,|\mathbf{p_2}|^2\,dp_2 = 4\pi\,|\mathbf{p_2}|^2\,dp_2\;. \tag{32}$$

Since $E_2 + E_3 = \sqrt{m_2^2c^4 + |\mathbf{p_2}|^2c^2} + \sqrt{m_3^2c^4 + |\mathbf{p_2}|^2c^2}$, one finds by differentiation:

$$d(E_2+E_3) = \frac{c^2\,|\mathbf{p_2}|\,dp_2}{\sqrt{m_2^2c^4+|\mathbf{p_2}|^2c^2}} + \frac{c^2\,|\mathbf{p_2}|\,dp_2}{\sqrt{m_3^2c^4+|\mathbf{p_2}|^2c^2}} = |\mathbf{p_2}|\,dp_2\left(\frac{1}{E_2}+\frac{1}{E_3}\right),\text{ or}$$

$$|\mathbf{p_2}|\,dp_2 = \frac{E_2E_3d(E_2+E_3)}{(E_2+E_3)c^2}\;. \tag{33}$$

Using Eqs. (32) and (33), Eq. (31) may be rewritten as

$$\begin{aligned}\Gamma &= \frac{Sc}{8\pi\hbar m_1}\int \frac{d(E_2+E_3)\,|\mathbf{p_2}|}{E_2+E_3}\delta(m_1c^2-E_2-E_3)\,|A|^2\\ &= \frac{S\,|\mathbf{p_2}|}{8\pi\hbar m^2c}\,|A|^2\;.\end{aligned} \tag{34}$$

The magnitude $|\mathbf{p_2}|$ of either outgoing momentum may be found from the conservation of energy,

$$m_1c^2 = \sqrt{m_2^2c^4+|\mathbf{p_2}|^2c^2} + \sqrt{m_3^2c^4+|\mathbf{p_2}|^2c^2}\;. \tag{35}$$

If $|A|^2$ is spin-dependent, it must be averaged over the possible initial particle spins and summed over the possible final particle spins. If there are multiple decay pathways, the total decay rate is the sum of the decay rates for each pathway, $\Gamma_{\text{Total}} = \sum\Gamma$. The time constant for particle decay is $\tau = 1/\Gamma$, and the particle half-life is $\tau_{1/2} = \tau\ln 2 = \ln 2/\Gamma$.

1.3.2 Pion Decay in the Toy Theory

If $m_\pi > 2m_n$ in the toy theory, the pion will be unstable and will decay into a nucleon-antinucleon pair. To lowest order, this process is described by a Feynman diagram of only one vertex and no internal lines, as shown in Fig. 3. Following the Feynman rules from the previous section, one finds

$$-iA(2\pi)^4\delta^4(p_1 - p_2 - p_3) = -ig(2\pi)^4\delta^4(p_1 - p_2 - p_3) \qquad \Longrightarrow \qquad A = g\ . \tag{36}$$

Solving Eq. (35) for the outgoing momentum using $m_1 = m_\pi$ and $m_2 = m_3 = m_n$, one finds

$$|\mathbf{p_2}| = |\mathbf{p_3}| = \frac{c}{2}\sqrt{m_\pi^2 - 4m_n^2}\ . \tag{37}$$

Note that this momentum is real only for $m_\pi > 2m_n$–otherwise the pion cannot decay. Plugging this momentum and the amplitude from Eq. (36) into Eq. (34), the decay rate is:

$$\Gamma = \frac{g^2\,|\mathbf{p_2}|}{8\pi\hbar m_\pi^2 c} = \frac{g^2\sqrt{m_\pi^2 - 4m_n^2}}{16\pi\hbar m_\pi^2}\ . \tag{38}$$

1.3.3 General Formula for Interaction Cross Sections

Now consider interactions between particles 1 and 2 in the center-of-mass frame: $1+2 \to 3+4+...$ The interaction rate is $\Gamma \propto \sigma\Delta v$, where σ is the interaction cross section and Δv is the collision velocity between particles 1 and 2. Thus the cross section $\sigma \propto \Gamma/\Delta v$ may be obtained by modifying Eq. (30) (where S is still a product of $1/N!$ for each N identical final particles):

$$\sigma \;=\; \frac{S\,c\hbar^2}{\Delta v}\underbrace{\left[\int\frac{d^3\mathbf{p_3}}{(2\pi)^3}\int\frac{d^3\mathbf{p_4}}{(2\pi)^3}...\right]}_{\substack{\text{Integrate over phase}\\ \text{space of final momenta}}}\underbrace{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4...)}_{\substack{\text{Conserve energy}\\ \text{\& momentum}}}\underbrace{\left(\frac{c}{2E_1}\frac{c}{2E_2}\frac{c}{2E_3}\frac{c}{2E_4}...\right)}_{\substack{\text{Normalize}\\ \text{fields}}}|A|^2\ . \tag{39}$$

The collision velocity may be rewritten using the velocities v_1 and v_2 of each particle:

$$\begin{aligned}\Delta v \;&=\; v_1 + v_2 \;=\; c^2\left(\frac{\gamma m_1 v_1}{\gamma m_1 c^2} + \frac{\gamma m_2 v_2}{\gamma m_2 c^2}\right)\\ &=\; c^2\,|\mathbf{p_1}|\left(\frac{1}{E_1} + \frac{1}{E_2}\right) \;=\; c^2\,|\mathbf{p_1}|\,\frac{(E_1+E_2)}{E_1E_2}\ .\end{aligned} \tag{40}$$

Using Eq. (40) and rewriting $\delta^4(p_1+p_2-p_3-p_4) = c\delta(E_1+E_2-E_3-E_4)\delta^3(-\mathbf{p_3}-\mathbf{p_4})$ (since $\mathbf{p_1}+\mathbf{p_2} = 0$ in the center-of-mass) σ for two final particles may be simplified just as Γ was earlier:

$$\begin{aligned}\sigma \;&=\; S\left(\frac{\hbar c}{8\pi}\right)^2\frac{c^2}{|\mathbf{p_1}|\,(E_1+E_2)}\int\frac{d^3\mathbf{p_3}d^3\mathbf{p_4}}{E_3E_4}\delta(E_1+E_2-E_3-E_4)\delta^3(-\mathbf{p_3}-\mathbf{p_4})\,|A|^2\\ &=\; S\left(\frac{\hbar c}{8\pi}\right)^2\frac{c^2}{|\mathbf{p_1}|\,(E_1+E_2)}\int\frac{d^3\mathbf{p_3}}{E_3E_4}\delta(E_1+E_2-E_3-E_4)\,|A|^2\\ &=\; S\left(\frac{\hbar c}{8\pi}\right)^2\frac{c^2}{|\mathbf{p_1}|\,(E_1+E_2)}\int\frac{d\Omega\,|\mathbf{p_3}|\,d(E_3+E_4)}{(E_3+E_4)}\delta(E_1+E_2-E_3-E_4)\,|A|^2\\ &=\; S\left(\frac{\hbar c}{8\pi}\right)^2\frac{|\mathbf{p_3}|}{|\mathbf{p_1}|}\frac{1}{(E_1+E_2)^2}\int d\Omega\,|A|^2\ .\end{aligned} \tag{41}$$

In simplifying the expression for Γ earlier, $\int d\Omega = 4\pi$ was used, since probabilities resulting from a stationary particle's decay were spherically symmetric (assuming $|A|^2$ was averaged over any initial particle spin). Here the integral over $d\Omega$ is left undone, because in general $|A|^2$ may depend on the angle between the initial and final particle trajectories. Thus the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2\frac{S}{(E_1+E_2)^2}\frac{|\mathbf{p}_{\text{final}}|}{|\mathbf{p}_{\text{initial}}|}\,|A|^2\ , \tag{42}$$

Differential cross section for two-body scattering in center-of-mass (CM) frame

where $|\mathbf{p}_{\text{initial}}|$ and $|\mathbf{p}_{\text{final}}|$ are the momenta of either particle before and after the collision.

For other conditions, modified versions of Eq. (42) may be derived using these same techniques. For example, for elastic scattering $1+2 \to 1'+2'$ between a massless particle 1 and a massive particle 2 in the lab frame where particle 2 is initially at rest, one finds

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_2 c}\right)^2 \left(\frac{E_1'}{E_1}\right)^2 \left\langle |A|^2 \right\rangle \tag{43}$$

Likewise, for the process $1+2 \to 3+4$ in the lab frame with particle 2 at rest and particles 3 and 4 massless, one should use

$$\frac{d\sigma}{d\Omega} = \frac{S}{2}\left(\frac{\hbar}{8\pi}\right)^2 \frac{c(E_1+m_2c^2)}{|\mathbf{p_1}|\,(E_1+m_2c^2-|\mathbf{p_1}|\,c\cos\theta)^2} \left\langle |A|^2 \right\rangle \tag{44}$$

1.3.4 Nucleon-Nucleon Scattering in the Toy Theory

An example of two-body interactions is nucleon-nucleon scattering in the toy theory. Applying the Feynman rules to the diagram in Fig. 4(a), one reproduces the results of Eqs. (28) and (29):

$$\begin{aligned} -iA_a(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') &= -ig^2(2\pi)^4 \int d^4p_0 \frac{1}{p_0^2-m_\pi^2c^2}\delta^4(p_1-p_0-p_1')\delta^4(p_2+p_0-p_2') \\ &= -ig^2(2\pi)^4 \frac{1}{(p_1-p_1')^2-m_\pi^2c^2}\delta^4(p_1+p_2-p_1'-p_2') \ , \end{aligned} \tag{45}$$

where integrating over the first δ^4 function yielded the substitution $p_0 = p_1 - p_1'$. (Using either δ^4 would produce an equivalent answer.) Thus the amplitude of the diagram in Fig. 4(a) is:

$$A_a = \frac{g^2}{(p_1-p_1')^2-m_\pi^2c^2} \ . \tag{46}$$

As shown in Fig. 4(b), there is actually a second diagram that contributes to this process. The only difference between diagrams (a) and (b) is that the two nucleons can “swap identities” with each other. For simplicity, this second diagram was swept under the rug in Section 1.2.3, but now it is crawling back out. Since the amplitude A_b of diagram (b) is simply the amplitude of (a) with the interchange $p_1' \leftrightarrow p_2'$, the total amplitude of this process to order g^2 is

$$A = A_a + A_b = \frac{g^2}{(p_1-p_1')^2-m_\pi^2c^2} + \frac{g^2}{(p_1-p_2')^2-m_\pi^2c^2} \ . \tag{47}$$

This collision is elastic, so in the center-of-mass frame $E_1 = E_1' = E_2 = E_2' \equiv E$ and $|\mathbf{p_1}| = |\mathbf{p_1'}| = |\mathbf{p_2}| = |\mathbf{p_2'}| \equiv |\mathbf{p}|$. Defining θ to be the angle between $\mathbf{p_1}$ and $\mathbf{p_1'}$, as shown in Fig. 4(c), one finds

$$(p_1-p_1')^2 = \frac{E_1^2+E_1'^2-2E_1E_1'}{c^2} - |\mathbf{p_1}|^2 - |\mathbf{p_1'}|^2 + 2\mathbf{p_1}\cdot\mathbf{p_1'} = -2\,|\mathbf{p}|^2\,(1-\cos\theta) \ . \tag{48}$$

Similarly, the other expression involving momenta is

$$(p_1-p_2')^2 = -2\,|\mathbf{p}|^2\,(1+\cos\theta) \ . \tag{49}$$

Using Eqs. (48) and (49), the amplitude from Eq. (47) may be rewritten as

$$A = -\frac{g^2}{2\,|\mathbf{p}|^2\,(1-\cos\theta)+m_\pi^2c^2} - \frac{g^2}{2\,|\mathbf{p}|^2\,(1+\cos\theta)+m_\pi^2c^2} \ . \tag{50}$$

Equation (42) may be used with $E^2 = |\mathbf{p}|^2\,c^2 + m_n^2c^4$ and the amplitude from Eq. (50):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{8E^2}\,|A|^2 \ . \tag{51}$$

1.4 Zoological Catalog of Fundamental Particles

Unlike in the toy theory, real particles have spin and are subject to four different fundamental forces, which will be considered individually in the next four sections. To avoid confusion, here are tables of the fundamental particles of matter and their interactions. Particles of matter may be divided into two families, leptons (Table 1) and quarks (Table 2). All are point-like, spin-$\frac{1}{2}$ particles. They all have corresponding antiparticles, denoted by adding an overbar (or sometimes for leptons by changing the sign of the charge). The four fundamental forces (Table 3) are caused by interactions with different force-mediating particles and act on different subsets of particles.

Charge	**First Generation**	**Second Generation**	**Third Generation**
-1	electron (e^-) 511.0 keV/c^2	muon (μ^-) 105.7 MeV/c^2	tau (τ^-) 1.784 GeV/c^2
0	electron neutrino (ν_e) < 3 eV/c^2	muon neutrino (ν_μ) < 190 keV/c^2	tau neutrino (ν_τ) < 18 MeV/c^2

Table 1. Leptons and their masses. μ^- and τ^- are basically short-lived, overweight electrons. Neutrinos are stable and have small but nonzero masses that have not yet been accurately measured. They do not interact by the electromagnetic or strong forces (or much via gravity), so they are difficult to detect and usually just seem like phantoms that carry off energy in weak interactions.

Charge	**First Generation**	**Second Generation**	**Third Generation**
-1/3	down (d) 310-363 MeV/c^2	strange (s) 483-538 MeV/c^2	bottom (b) 4700 MeV/c^2
+2/3	up (u) 310-363 MeV/c^2	charm (c) 1500 MeV/c^2	top (t) $> 23,000$ MeV/c^2

Table 2. Quarks and their effective masses. Baryons like neutrons and protons are composed of three quarks, and mesons like pions and kaons are composed of a quark and an antiquark. The possible quark **flavors** (types) are d, u, c, s, b, and t. The effective masses of the quarks in mesons and baryons is given; where a range of mass values is given, the lower value is in mesons and the upper one is in baryons. d and u quarks are commonplace, s quarks can be important at times, and c, b, and t quarks are so massive that they are important only in very high-energy collisions.

Force	**Particle**	**Spin**	**Charge**	**Mass**	**Interacts with**
Electromagnetic	photon	1	0	0	all charged particles
Weak nuclear	$W^\pm$	1	±1	81.8 GeV/c^2	all leptons and quarks
	Z^0	1	0	92.6 GeV/c^2	all leptons and quarks
Strong nuclear	gluons	1	0	0	other gluons and all quarks
Gravitational	graviton	2	0	0	all particles

Table 3. Fundamental forces and the particles that mediate them. The physical roles of the electromagnetic and gravitational forces should be familiar. The weak nuclear force mediates some important particle decay processes such as beta decay of neutrons and muons. The strong nuclear force binds together quarks to form baryons and mesons; in turn, mesons (or really the underlying quark-gluon strong force interactions) effectively act as force-mediating particles to bind together baryons to form the nuclei of atoms.

2 Electromagnetic Force: Quantum Electrodynamics

Quantum electrodynamics (QED) describes electromagnetic interactions between an electrically charged particle and a photon. It also describes the electromagnetic force between two charged particles, since this force is mediated by the interchange of virtual photons. QED is similar to the toy field theory discussed in the previous section, but it is more complicated, as it must include the effects of particle spin: photons have spin 1, and most charged particles of interest have spin $\frac{1}{2}$. QED may be used to correctly calculate many physical effects, such as Compton scattering of a photon off an electron, electron-positron annihilation, scattering of an electron off a proton or another electron, and the interaction of an electron with a magnetic field.

2.1 Quantum Electrodynamics (QED) Theory

The Feynman rules for QED may be derived just as they were for the toy theory of Section 1. Due to the many similarities between QED and the toy theory, we will focus on the key difference, particle spin, and the impact this has on the Feynman rules. Spin-$\frac{1}{2}$ charged particles such as electrons will be considered first, and then spin-1 photons will be analyzed. Next the interactions between the two types of particles will be considered and the Feynman rules will be discussed. Subsequently these rules will be applied to several practical examples of physical effects.

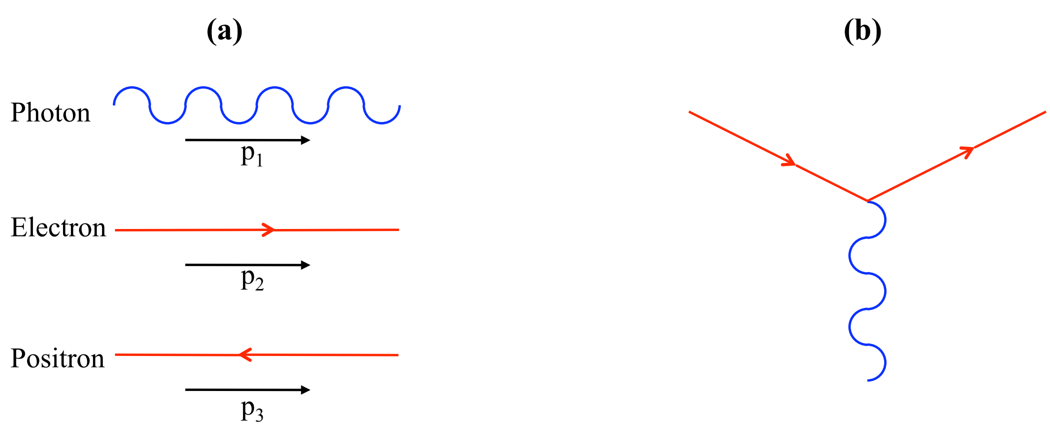

Figure 5. Particles and their interactions in QED Feynman diagrams. (a) Wavy lines represent photons, lines with arrows pointing the same way as their momentum (forward in time) represent electrons, and lines with arrows pointing opposite their momentum (backward in time) represent positrons. **(b)** A QED interaction vertex indicates an incoming or outgoing photon interacting with an incoming electron (or outgoing positron) and outgoing electron (or incoming positron).

2.1.1 Spin-1/2 Particle Field: Dirac Equation

For spin-$\frac{1}{2}$ particles, the Klein-Gordon equation may be "factored" into two separate, spin-dependent solutions. As shown in Section 1, the Klein-Gordon equation is basically just a fancy version of the relativistic energy-momentum relation. Therefore, factoring the Klein-Gordon equation is equivalent to factoring the momentum-energy relationship: $p^2 - m^2c^2 = (p_\mu - mc)(p^\mu + mc) = 0$. However, factoring like this wouldn't be quite right. Although p^2 and m^2c^2 are scalars, when they are not squared, p_μ is a four-vector and mc is a scalar. Trying to directly add them together in the factored terms is like adding apples and oranges and is sure to bring out the math cops. This problem can be fixed by introducing the 4×4 Dirac matrices γ^μ that make the factoring work properly:

$$(p^2 - m^2c^2) \;\;=\;\; (\gamma^\mu p_\mu - mc)\,(\gamma^\nu p_\nu + mc) \;=\; 0\,. \tag{52}$$

Now $\gamma^\mu p_\mu$ and $\gamma^\nu p_\nu$ are scalars (using the Einstein summation convention for repeated indices from *Relativity* ?.?), so everything is legal. Multiplying four-vectors by Dirac matrices is so common that it is usually denoted with a slash mark, $\not{a} \equiv \gamma^\mu a_\mu$ for any four-vector a_μ.

The momentum of a field obeys the usual quantum mechanical relationship, $p_\mu \psi = i\hbar\psi$, so the two factored solutions in Eq. (52) are

$$(i\hbar\,\not{\partial} - mc)\psi \;=\; 0 \qquad \textbf{Dirac equation for spin-}\tfrac{1}{2}\textbf{ particles (e.g., electrons)} \tag{53}$$

$$(i\hbar\,\not{\partial} + mc)\psi \;=\; 0 \qquad \textbf{Dirac equation for spin-}\tfrac{1}{2}\textbf{ antiparticles (e.g., positrons)} \tag{54}$$

To satisfy $\gamma^\mu p_\mu \gamma^\nu p_\nu = p^2$ in Eq. (52) work, the Dirac matrices must obey certain properties:

$$\left.\begin{array}{r} (\gamma^0)^2 = 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \text{ for } \mu \neq \nu \end{array}\right\} \implies \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{55}$$

Appropriate 4×4 matrices are

$$\gamma^0 \equiv \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \quad \text{and} \quad \gamma^i \equiv \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix} \text{ for } i = 1, 2, 3, \tag{56}$$

where the 2×2 identity ($\mathbf{1}$) and zero ($\mathbf{0}$) matrices are defined as

$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{57}$$

and the 2×2 Pauli spin matrices are the same as in nonrelativistic quantum physics,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{58}$$

Like all the four-vectors and 4×4 matrices, the Dirac solutions ψ also have four components,

$$\psi \;=\; \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad\qquad \overline{\psi} \;\equiv\; \psi^\dagger\gamma^0 \;=\; (\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*)\,. \tag{59}$$

If ψ represents an incoming particle, its **adjoint** $\overline{\psi}$ represents an outgoing particle of the same type, and vice versa. $\psi^\dagger$ denotes the Hermitian (transpose) conjugate as in nonrelativistic quantum physics and must be multiplied by γ^0 to obtain the appropriate minus signs for $\overline{\psi}$. From now on, a bar over any field solution will denote this same adjoint-taking process.

The general idea in Eq. (59) is that ψ_1 represents an electron with one spin direction (say spin up), ψ_2 is an electron with the other spin direction (say down), ψ_3 is a positron with one spin direction, and ψ_4 is a positron with the other spin direction. Thus, one can write out the solutions for incoming electrons or outgoing positrons with the first or second spin state and normalization factor C,

$$\begin{aligned} \psi_{\text{electron}} &= C\, e^{-ip\cdot x/\hbar} u_s \\ \psi_{\text{positron}} &= C\, e^{+ip\cdot x/\hbar} v_s \end{aligned} \qquad u_1 \sim \begin{pmatrix} 1 \\ 0 \\ - \\ - \end{pmatrix} \quad u_2 \sim \begin{pmatrix} 0 \\ 1 \\ - \\ - \end{pmatrix} \quad v_1 \sim \begin{pmatrix} - \\ - \\ 1 \\ 0 \end{pmatrix} \quad v_2 \sim \begin{pmatrix} - \\ - \\ 0 \\ 1 \end{pmatrix} \tag{60}$$

For mathematical reasons [1], the blanks in the column vectors are not actually zero, but they are completely determined by the choices of the other values in the column and thus may be ignored.

Outgoing electrons or incoming positrons are represented by taking the adjoints of Eq. (60):

$$\overline{\psi_{\text{electron}}} = C^*\, e^{+ip\cdot x/\hbar} \overline{u_s} \qquad \overline{\psi_{\text{positron}}} = C^*\, e^{-ip\cdot x/\hbar} \overline{v_s}\ . \tag{61}$$

The "electron" and "positron" subscripts will be dropped, since the particles are distinguished by u vs. v Dirac solutions. The u and v solutions have the following properties:

$$\textbf{Orthogonality} \qquad \overline{u_1} u_2 = u_1 \overline{u_2} = 0 \qquad \overline{v_1} v_2 = v_1 \overline{v_2} = 0 \tag{62}$$

$$\textbf{Normalization} \qquad \overline{u_1} u_1 = u_2 \overline{u_2} = 2mc \qquad \overline{v_1} v_1 = v_2 \overline{v_2} = -2mc \tag{63}$$

$$\textbf{Completeness} \qquad \overline{u_1} u_1 + \overline{u_2} u_2 = \not{p} + mc \qquad \overline{v_1} v_1 + \overline{v_2} v_2 = \not{p} - mc \tag{64}$$

Written in terms of momentum, the Dirac equation is $[\not{p} - mc]\psi = 0$; note the quantity in brackets. Using Eq. (27), the propagator for a virtual particle obeying the Dirac equation is i times the (matrix) inverse of this quantity in brackets

$$\frac{i}{\not{p} - mc} = i \frac{\not{p} + mc}{p^2 - m^2 c^2} \qquad \textbf{propagator for virtual electron}\ . \tag{65}$$

The propagator in Eq. (65) was rewritten simply to get the γ^μ in $\not{p}$ out of the denominator.

A virtual electron going one way is the same as a virtual positron going the opposite direction in space and time. Either one leads to the same propagator. Physically, a reaction that requires a virtual particle will get a propagator in its amplitude (as in nonrelativistic quantum physics, amplitude squared is proportional to the probability that something will happen). A particle of mass m must have an energy of at least mc^2 to be real. Virtual particles do not have enough energy to be real (Section 1.1.2). The further a virtual particle is from having enough energy to officially exist, the smaller its propagator is and hence the smaller the probability that a reaction requiring such a virtual particle will occur.

The Dirac equation may be derived from the Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\hbar c\ \not{\partial} - mc^2)\psi\ . \tag{66}$$

2.1.2 Spin-1 Massless Particle Field: Photon Equation

As discussed in the *Electromagnetism*, electromagnetic fields may be described in terms of different variables and notations. Conceptually the simplest variables are just the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, which are physical quantities that can be measured directly. To make the notation more compact, the electric and magnetic fields can be expressed as derivatives (with respect to space and time) of the scalar potential Φ and vector potential $\mathbf{A}$, which themselves are not directly measurable:

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial A}{\partial t} \tag{67}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{68}$$

Since all that matters is the derivatives of Φ and $\mathbf{A}$, one can add arbitrary constants to Φ and $\mathbf{A}$ without affecting the actual physics. Different gauges are basically different conventions for defining the precise values of Φ and $\mathbf{A}$. All gauges yield the same physically measurable $\mathbf{E}$ and $\mathbf{B}$ in the end, but various gauges make different calculations easier to do. Lorentz gauge will be chosen in order to simplify the calculation below, whereas Coulomb gauge will be chosen to simplify the calculation in Section 2.2.1. For more information on potentials, gauges, and other notation, see *Electromagnetism* ?.?.

The scalar and vector potentials may be combined into a four-vector electromagnetic potential $A^\mu = (\Phi,\ \mathbf{A})$. Likewise, the charge density ρ and current density $\mathbf{J}$ can form a four-vector electric current density, $J^\mu = (c\rho,\ \mathbf{J})$. The notation may be modified even further to introduce an electromagnetic field strength tensor $F^{\mu\nu}$:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{69}$$

Using this notation, Maxwell's equations may be written very compactly as

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad \textbf{Maxwell's equations} \tag{70}$$

In the Lorentz gauge, $\partial_\mu A^\mu = 0$, Maxwell's equations may be rewritten as

$$\frac{4\pi}{c} J^\nu = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial^2 A^\nu \ . \tag{71}$$

If there are no sources, $J^\nu = 0$, one obtains the equation for a free electromagnetic field,

$$\partial^2 A^\nu = 0 \qquad \textbf{Free electromagnetic field (Lorentz gauge)} \tag{72}$$

Making the quantum mechanical substitution $i\hbar\partial_\mu \to p_\mu$ and explicitly writing in the metric, Eq. (72) may be rewritten as a momentum-space equation:

$$\left[-p^2 g_{\mu\nu}\right] A^\nu = 0 \qquad \textbf{Free electromagnetic field momentum eq. (Lorentz gauge)} \tag{73}$$

From Eq. (27), the propagator for a photon of an electromagnetic field is i times the (tensor) inverse of the bracketed quantity in Eq. (73) (using $g^{-1}_{\mu\nu} = g_{\mu\nu}$):

$$-i\frac{g_{\mu\nu}}{p^2} \qquad \textbf{Photon propagator (Lorentz gauge)} \tag{74}$$

Maxwell's equations may be derived from the Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^\mu A_\mu \ . \tag{75}$$

A^μ solutions for photons coming in or out of an interaction may be written in the form

$$A^\mu_{\text{incoming}} = a\epsilon^\mu_{(s)} e^{-ip\cdot x/\hbar} \qquad A^\mu_{\text{outgoing}} = a\epsilon^{\mu\,*}_{(s)} e^{+ip\cdot x/\hbar} \ , \tag{76}$$

where a is the amplitude and the photon four-momentum is simply $p = (|\mathbf{p}|\,, \mathbf{p})$. The polarization vector $\epsilon_{(s)}$ denotes one of two possible photon polarizations $\epsilon^\mu_{(1)}$ and $\epsilon^\mu_{(2)}$ that are orthogonal to each other ($\epsilon^{\mu\,*}_{(1)}\epsilon_{(2)\,\mu} = 0$) and normalized ($\epsilon^{\mu\,*}_{(s)}\epsilon_{(s)\,\mu} = 1$). For example, for photons going in the $\hat{\mathbf{z}}$ direction, one could choose $\epsilon^\mu_{(1)} = \hat{\mathbf{x}}$ and $\epsilon^\mu_{(2)} = \hat{\mathbf{y}}$. Note that the Lorentz gauge condition $\partial_\mu A^\mu = 0$ implies that $p_\mu \epsilon^\mu_{(s)} = 0$.

2.1.3 Interactions and Feynman Rules for QED

The electric current density J^μ that interacted with photons in Section 2.1.2 is the same as the electric current density of electron or positron fields:

$$J^\mu = ce\overline{\psi}\gamma^\mu\psi \,. \tag{77}$$

Adding together the Lagrangians for electrons/positrons and photons from Eqs. (66) and (75) and making the substitution from Eq. (77), one obtains the total Lagrangian for QED:

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(i\hbar c\ \not{\partial} - mc^2)\psi - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - e\overline{\psi}\gamma^\mu\psi A_\mu \,. \tag{78}$$

The final term in Eq. (78) represents interactions between charged particles and photons. Inserting it into Eq. (17) yields the electromagnetic coupling constant g_e between particles of charge e and photons,

$$g_e \ \equiv\ \sqrt{\frac{4\pi}{\hbar c}}e \ =\ \sqrt{4\pi\alpha} \ \approx\ 0.303 \,, \tag{79}$$

where $\alpha \equiv e^2/\hbar c \approx 1/137$ is called the fine structure constant.

Based on the derivations above and in Sections 2.1.1 and 2.1.2, the Feynman rules for QED (compared with those for the spinless toy theory in Section 1.2.3) are:

1. Label the four-momentum p_i of each line, using an arrow to denote the positive direction. Also note the spin on all external lines. Then multiply the relevant following factors together, starting at the last fermion line in the diagram and working tip-to-tail backwards through the diagram.

2. For each initial or final particle (external line), include the appropriate factor:

Incoming electron: u	Outgoing electron: $\overline{u}$
Incoming positron: $\overline{v}$	Outgoing positron: v
Incoming photon: ϵ^μ	Outgoing photon: $\epsilon^{\mu\,*}$

These factors are basically the wavefunctions of initial and final particles, and they account for the momentum and polarization of those particles.

3. For each virtual particle (internal line), integrate over all possible four-momenta with a weighting factor (propagator) that makes momenta further from the particle's mass shell less likely to occur:

$$\int \frac{d^4p_i}{(2\pi)^4}\,\frac{i(\not{p}_i + m_i c)}{p_i^2 - m_i^2c^2} \qquad \text{for virtual fermions}$$

$$\int \frac{d^4p_i}{(2\pi)^4}\,\frac{-ig_{\mu\nu}}{p_i^2} \qquad \text{for virtual photons}$$

where p_i and m_i are the four-momentum and mass of that virtual particle.

4. For each interaction vertex, include a factor

$$(ig_e\gamma^\mu)(2\pi)^4\delta^4(\textstyle\sum p_{\text{in}} - \sum p_{\text{out}}) \,,$$

in which p_{in} and p_{out} are the momenta entering and leaving the vertex. The electromagnetic interaction strength is $g_e \equiv \sqrt{4\pi\alpha}$ and the δ^4 function ensures conservation of energy and momentum.

5. For each closed fermion loop, include a factor of -1 and take the trace.

6. Defining the initial and final momenta as p_{initial} and p_{final}, the net result is

$$iA(2\pi)^4\delta^4(\textstyle\sum p_{\text{initial}} - \sum p_{\text{final}}) \,.$$

7. If two Feynman diagrams that contribute to the same process differ only by the interchange of two identical fermion lines, multiply one of their amplitudes by -1 when the amplitudes are summed. This is simply due to fermions acting like fermions.

As in Section 1, the probability of a process happening is proportional to $|\sum A|^2$. If $|A|^2$ is spin-dependent, it must be averaged over the possible initial particle spins and summed over the possible final particle spins, as noted in Section 1. Then decays may be computed using Eqs. (34) and (35) and interactions may be calculated using Eq. (42).

There are several rules which are very handy for evaluating Feynman diagrams. The first, called **Casimir's trick**, is useful for summing over the spins s_a and s_b of particles a and b:

$$\sum_{s_a}\sum_{s_b}\left[\overline{u}(a)\Gamma_1 u(b)\right]\left[\overline{u}(a)\Gamma_2 u(b)\right]^* = \sum_{s_a}\sum_{s_b}\left[\overline{u}(a)\Gamma_1 u(b)\right]\left[\overline{u}(b)\overline{\Gamma_2}u(a)\right]$$

$$= \sum_{s_a}\overline{u}(a)\Gamma_1\left(\not{p}_b + m_b c\right)\overline{\Gamma_2}u(a) = \mathrm{Tr}\left[\Gamma_1\left(\not{p}_b + m_b c\right)\overline{\Gamma_2}\left(\not{p}_a + m_a c\right)\right] , \tag{80}$$

where Tr denotes the trace of the matrix and $\overline{\Gamma_2} \equiv \gamma^0\Gamma_2^\dagger\gamma^0$. If either u on the left side of Eq. (80) is replaced with a v, the corresponding mass on the right side changes sign.

Since Casimir's trick reduces the evaluation of Feynman diagrams to multiplying matrices and taking traces, here is a short catalog of useful matrix multiplication and trace theorems:

Basic properties:

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \tag{81}$$

$$\mathrm{Tr}(\alpha A) = \alpha\mathrm{Tr}(A) \tag{82}$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA) \neq \mathrm{Tr}(\text{noncyclic permutations}) \tag{83}$$

Contraction theorems:

$$g_{\mu\nu}g^{\mu\nu} = 4 \tag{84}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \rightarrow \not{a}\not{b}+\not{b}\not{a} = 2a\cdot b \tag{85}$$

$$\gamma_\mu\gamma^\mu = 4 \tag{86}$$

$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \rightarrow \gamma_\mu \not{a}\gamma^\mu = -2\not{a} \tag{87}$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda} \rightarrow \gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b \tag{88}$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu \rightarrow \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} \tag{89}$$

Trace theorems:

$$\mathrm{Tr}(\text{product of odd number of }\gamma\text{ matrices:}) = 0 \tag{90}$$

$$\mathrm{Tr}(1) = 4 \tag{91}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \rightarrow \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b \tag{92}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \rightarrow \mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4(a{\cdot}b c{\cdot}d - a{\cdot}c b{\cdot}d + a{\cdot}d b{\cdot}c) \tag{93}$$

$$\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3...\not{p}_{2n}) = p_1\cdot p_2\mathrm{Tr}(\not{p}_3...\not{p}_{2n}) - p_1\cdot p_3\mathrm{Tr}\left(\not{p}_2\not{p}_4...\not{p}_{2n}\right) + ... + p_1\cdot p_{2n}\mathrm{Tr}(\not{p}_2\not{p}_3...\not{p}_{2n-1}) \tag{94}$$

Trace theorems with $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$:

$$\mathrm{Tr}(\text{product of }\gamma^5\text{ \& odd no. of }\gamma\text{ matrices}) = 0 \tag{95}$$

$$\mathrm{Tr}(\gamma^5) = 0 \tag{96}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0 \rightarrow \mathrm{Tr}(\gamma^5\not{a}\not{b}) = 0 \tag{97}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma} \rightarrow \mathrm{Tr}(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma , \tag{98}$$

$$\text{where } \epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1 & \text{if } \mu\nu\lambda\sigma \text{ has an even no. of inter-} \\ & \text{changes of indices from 0123} \\ +1 & \text{if odd no. of interchanges} \\ 0 & \text{if any two indices are the same} \end{cases} \tag{99}$$

2.2 Examples

Now we will use the QED Feynman rules to calculate a number of physical processes ranging from Compton scattering to the anomalous magnetic moment of the electron.

2.2.1 Compton Scattering

Compton scattering is scattering of a photon off of an electron. Experimental measurement of this process was particularly important in the very early days of quantum physics, since it showed that light could behave as particles or quanta instead of just waves as predicted by Maxwell's equations.

The lowest-order contributions to Compton scattering come from the Feynman diagrams in Fig. 6(a) and (b), and Fig. 6(c) shows the scattering process as it actually appears in physical space. The electron has initial four-momentum p (chosen to be at rest in the lab frame) and final momentum p'. The initial and final momenta of the photon are q and q', respectively. Due to the collision, the photon is scattered by an angle θ from its initial trajectory.

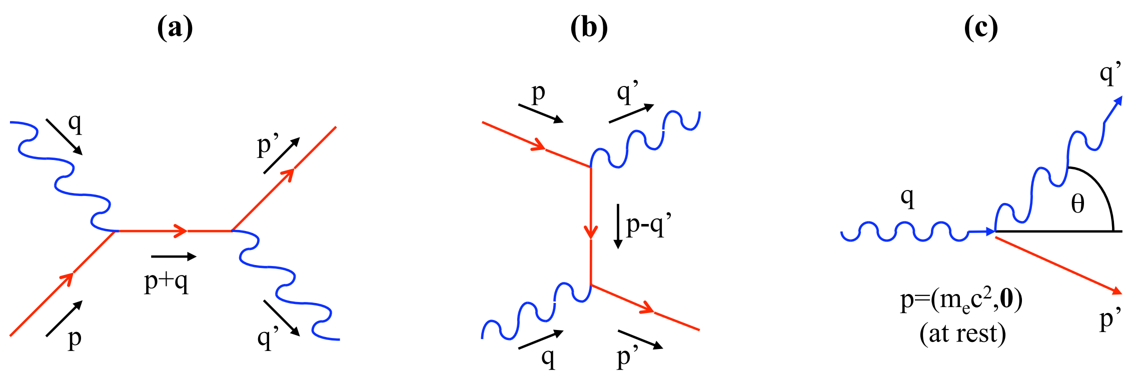

Figure 6. Compton scattering. (a) and **(b)** Lowest-order Feynman diagrams contributing to the process. Note that (b) looks like (a) rotated clockwise by 90^o. **(c)** Photon scattering angle in the lab frame (electron initially at rest).

Conservation of momentum and energy for Fig. 6(c) can be used to express the final momentum of the photon simply in terms of the initial photon momentum and the photon scattering angle θ. Momentum conservation yields the relation

$$\mathbf{p}' = \mathbf{q} - \mathbf{q}' \quad \Longrightarrow \quad |\mathbf{p}'|^2 \ = \ |\mathbf{q}|^2 - 2\,|\mathbf{q}|\,|\mathbf{q}'|\cos\theta + |\mathbf{q}'|^2 \tag{100}$$

The equation of energy conservation (divided by c) for Fig. 6(c) is

$$|\mathbf{q}| + m_e c \ = \ |\mathbf{q}| + \sqrt{|\mathbf{p}'|^2 + m_e^2 c^2} \quad \Longrightarrow \quad (m_e c + |\mathbf{q}| - |\mathbf{q}'|)^2 \ = \ |\mathbf{p}'|^2 + m_e^2 c^2 \tag{101}$$

Equation (100) may be substituted into Eq. (101) to eliminate $|\mathbf{p}'|^2$ and solve for $|\mathbf{q}'|$:

$$|\mathbf{q}'| \ = \ \frac{|\mathbf{q}|}{1 + \frac{|\mathbf{q}|}{m_e c}(1-\cos\theta)} \quad \Longrightarrow \quad \omega' \ = \ \frac{\omega}{1 + \frac{\hbar\omega}{m_e c}(1-\cos\theta)} \tag{102}$$

The relation in Eq. (102) was rewritten in terms of the initial and final angular frequencies of the photon, $\omega \equiv |\mathbf{q}|\,c/\hbar$ and $\omega' \equiv |\mathbf{q}'|\,c/\hbar$. As promised, the final photon momentum (or frequency) simply depends on the initial photon momentum (or frequency) and the scattering angle.

The next step is to calculate the differential cross section so that we will know how likely the photon is to be scattered by a certain angle. This is a long calculation and makes use of all the QED rules and tricks we have developed, so strap yourself in for a ride. Using the Feynman rules and defining ϵ and ϵ' as the initial and final polarizations of the photon, the amplitude for Fig. 6(a) is

$$\begin{aligned}
A_a &= \frac{g_e^2\,\overline{u}(p')\,\not\epsilon'(\not p+\not q+m_e c)\,\not\epsilon u(p)}{(p+q)^2-m_e^2c^2} \\
&= \frac{g_e^2\,\overline{u}(p')\,\not\epsilon'(\not p+\not q+m_e c)\,\not\epsilon u(p)}{2p\cdot q} && \text{(using } p^2=m_e^2c^2 \text{ and } q^2=0) \\
&= \frac{g_e^2\,\overline{u}(p')\,\not\epsilon'\,\not\epsilon(2p\cdot\epsilon+2q\cdot\epsilon-\not p-\not q+m_e c)u(p)}{2p\cdot q} && \text{[using Eq. (85)]} \\
&= -\frac{g_e^2\,\overline{u}(p')\,\not\epsilon'\,\not\epsilon\,\not q u(p)}{2p\cdot q}\,, && (103)
\end{aligned}$$

The last step in Eq. (103) used the Coulomb gauge ($q{\cdot}\epsilon=0$), the freedom to choose the gauge and frame such that $p{\cdot}\epsilon=0$, and the Dirac equation $[(\not p-m_e c)u(p)=0]$.

The diagram in Fig. 6(b) looks like the one in 6(a) rotated clockwise by 90^o. Amplitudes for Feynman diagrams are the same regardless of their orientation, provided that appropriate variable names are changed. This important rule is called **crossing symmetry**. Thus the amplitude A_b for Fig. 6(b) may be obtained from A_a by swapping the appropriate variables, $\epsilon\leftrightarrow\epsilon'$ and $q\leftrightarrow -q'$:

$$A_b = -\frac{g_e^2\,\overline{u}(p')\,\not\epsilon\,\not\epsilon'\,\not q' u(p)}{2p\cdot q'}\,. \tag{104}$$

Adding the two amplitudes yields the total amplitude for Compton scattering to lowest order:

$$A = A_a + A_b = -g_e^2\overline{u}(p')\Gamma u(p)\,, \qquad \text{where}\quad \Gamma \equiv \frac{\not\epsilon'\,\not\epsilon\,\not q}{2p\cdot q}+\frac{\not\epsilon\,\not\epsilon'\,\not q'}{2p\cdot q'}\,. \tag{105}$$

Note that $\overline{\Gamma}$ is the same as Γ except with the matrix order reversed.

Averaging over the initial electron spin (introducing a factor of $\frac{1}{2}$) and summing over the final electron spin yields

$$\begin{aligned}
\left\langle |A|^2\right\rangle &= \frac{1}{2}\sum_{\text{spins}} g_e^4\left|\overline{u}(p')\Gamma u(p)\right|^2 = \frac{g_e^4}{2}\,\mathrm{Tr}\left[\Gamma\,(\not p+m_e c)\,\overline{\Gamma}\,(\not p'+m_e c)\right] && \text{[using Eq. (80)]} \\
&= \frac{g_e^4}{2}\,\mathrm{Tr}\left[\left(\frac{\not\epsilon'\,\not\epsilon\,\not q}{2p\cdot q}+\frac{\not\epsilon\,\not\epsilon'\,\not q'}{2p\cdot q'}\right)(\not p+m_e c)\left(\frac{\not q\,\not\epsilon\,\not\epsilon'}{2p\cdot q}+\frac{\not q'\,\not\epsilon'\,\not\epsilon}{2p\cdot q'}\right)(\not p'+m_e c)\right] \\
&= \frac{g_e^4}{2}\,\left(\mathrm{Tr}_1+2\mathrm{Tr}_2+\mathrm{Tr}_3\right)\,. && (106)
\end{aligned}$$

The component traces in Eq. (106) are defined below and may be simplified using Eq. (94) and

$$\begin{aligned}
\text{Particles on mass shell} &\implies q^2=q'^2=0 \qquad p^2=p'^2=m_e^2c^2 \\
\text{Choice of gauge} &\implies \epsilon\cdot q=\epsilon'\cdot q'=0 \qquad \epsilon\cdot p=\epsilon'\cdot p=0 \\
\text{Photon polarization normalized} &\implies \epsilon^2=\epsilon'^2=-1 \\
p+q=p'+q' \;\rightarrow\; (p-q')^2=(p'-q)^2 &\implies p\cdot q'=p'\cdot q \\
p+q=p'+q' \;\rightarrow \epsilon'\cdot(p+q)=\epsilon'\cdot(p'+q') &\implies \epsilon'\cdot q=\epsilon'\cdot p'
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_1 &\equiv \frac{1}{(2p\cdot q)^2}\mathrm{Tr}[\not\epsilon'\not\epsilon\not q(\not p+m_e c)\not q\not\epsilon\not\epsilon'(\not p'+m_e c)] \;=\; \frac{1}{2p\cdot q}\mathrm{Tr}[\not\epsilon'\not\epsilon\not q\not\epsilon\not\epsilon'\not p'] \\
&= \frac{1}{2p\cdot q}\mathrm{Tr}[\not\epsilon'\not q\not\epsilon'\not p'] \;=\; \frac{2p\cdot q'+4(\epsilon'\cdot q)^2}{p\cdot q} \qquad (107)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_2 &\equiv \frac{1}{4(p\cdot q)(p\cdot q')}\mathrm{Tr}[\not\epsilon'\not\epsilon\not q(\not p+m_e c)\not q'\not\epsilon'\not\epsilon(\not p'+m_e c)] \\
&= \frac{\mathrm{Tr}[\not q'\not\epsilon'\not\epsilon\not\epsilon'\not\epsilon\not p]}{2p\cdot q'}+\frac{2(q'\cdot\epsilon)^2}{p\cdot q'}-\frac{2(q\cdot\epsilon')^2}{p\cdot q} \\
&= 4(\epsilon'\cdot\epsilon)^2-2+\frac{2(q'\cdot\epsilon)^2}{p\cdot q'}-\frac{2(q\cdot\epsilon')^2}{p\cdot q} \qquad (108)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_3 &\equiv \frac{1}{(2p\cdot q')^2}\mathrm{Tr}[\not\epsilon\not\epsilon'\not q'(\not p+m_e c)\not q'\not\epsilon'\not\epsilon(\not p'+m_e c)] \\
&= \begin{array}{c}\mathrm{Tr}_1\text{ with }\epsilon\leftrightarrow\epsilon'\\ \text{and }q\leftrightarrow -q'\end{array} \;=\; \frac{2p\cdot q-4(\epsilon\cdot q')^2}{p\cdot q'} \qquad (109)
\end{aligned}$$

Some terms cancel out when the traces are added together, leaving

$$\left\langle |A|^2\right\rangle \;=\; g_e^4\left[\frac{p\cdot q'}{p\cdot q}+\frac{p\cdot q}{p\cdot q'}+4(\epsilon'\cdot\epsilon)^2-2\right] \;=\; g_e^4\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+4(\epsilon'\cdot\epsilon)^2-2\right], \qquad (110)$$

where the last step used the lab frame in which $p=(m_e c,\mathbf{0})$ and thus $p\cdot q=m_e\hbar\omega$ and $p\cdot q'=m_e\hbar\omega'$. The recurring ratio ω'/ω may be found from Eq. (102).

Inserting Eq. (110) into Eq. (43), one obtains

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol.}} \;=\; \frac{1}{4}r_e^2\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+4(\epsilon'\cdot\epsilon)^2-2\right] \qquad \textbf{Polarized Klein-Nishina formula for Compton scattering} \qquad (111)$$

where $r_e\equiv e^2/(m_e c^2)\approx 2.8\times10^{-15}$ m is called the classical radius of the electron, since in low-energy collisions the electron seems to have a radius of that order, as is becoming manifest in this calculation. Of course, the electron is really a point particle, and this apparent radius is simply a measure of the "effective reach" of the electric field surrounding the electron.

One can sum over the initial photon polarization $\epsilon=\epsilon_1$ or ϵ_2 and the final polarization $\epsilon'=\epsilon_1'$ or ϵ_2'. One is free to define the coordinate axes to correspond to the initial polarizations, so that $\epsilon_1=\hat{\mathbf{x}}$ and $\epsilon_2=\hat{\mathbf{y}}$. After being scattered by an angle θ as in Fig. 6(c), the outgoing photon has the possible polarization states $\epsilon_1'=\hat{\mathbf{x}}$ and $\epsilon_2'=\cos\theta\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}}$. Thus the sum over initial and final photon polarizations is

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(\epsilon_i'\cdot\epsilon_j)^2 \;=\; 1+\cos^2\theta \;=\; 2-\sin^2\theta\,. \qquad (112)$$

Averaging over the initial photon polarization (using a factor of $\frac{1}{2}$) and summing over the final polarization, the complete expression for the differential cross section becomes

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol.}} &= \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol.}} \\
&= \frac{1}{2}r_e^2\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right) \qquad \textbf{Unpolarized Klein-Nishina formula} \qquad (113)
\end{aligned}$$

If the photon energy is not changed much by the collision, $\omega' \approx \omega$, Eqs. (111) and (113) reduce to the differential cross section for classical Thomson scattering of a photon by an electron,

$$\frac{d\sigma}{d\Omega} = r_e^2(\epsilon' \cdot \epsilon)^2 \qquad \textbf{for polarized Thomson scattering} \tag{114}$$

$$= \frac{1}{2} r_e^2 (1 + \cos^2\theta) \qquad \textbf{for unpolarized Thomson scattering} \tag{115}$$

Equation (115) may easily be integrated to obtain the total cross section,

$$\sigma = \int_0^\pi (d\theta\, 2\pi \sin^2\theta) \frac{1}{2} r_e^2 (1+\cos^2\theta) = \frac{8\pi}{3} r_e^2 \qquad \textbf{Thomson cross section} \tag{116}$$

Of course, the exact Klein-Nishina cross section can also be integrated from Eq. (113), but the result is more mathematically frightening than physically enlightening.

2.2.2 Electron-Positron Annihilation

As mentioned in Section 1, particles of matter and antimatter can annihilate each other. As an example, an electron and a positron can annihilate each other, converting their total energy into two photons. The two lowest-order Feynman diagrams contributing to this process are shown in Fig. 7 (a) and (b); these diagrams are identical except that the two outgoing photons are swapped. The incoming electron and positron have four-momenta p and p' respectively, and the outgoing photons have momenta q and q' and polarizations ϵ and ϵ'.

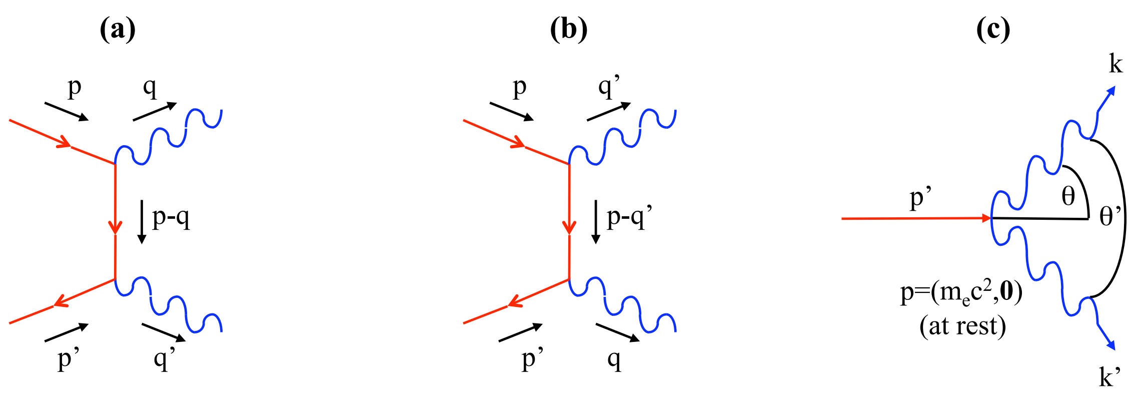

Figure 7. Pair annihilation of an electron and a positron into two photons. (a) and **(b)** Lowest-order Feynman diagrams contributing to the process. The only difference between the two diagrams is that the identities of the two photons have been interchanged. Note that these diagrams look the same as Fig. 6(b) with the bottom half twisted. **(c)** Angle of photon emission relative to the incoming positron trajectory in the lab frame (electron initially at rest).

The simplest way to calculate the cross section for electron-positron annihilation is to note the similarity of the diagrams in Fig. 7 to those in Fig. 6 for Compton scattering. As has been explained, crossing symmetry dictates that the amplitude remains the same if the diagrams look the same. In this case, one can use the amplitude from Eq. (110), make the change $q \to -q$ since the momentum of both photons is now outgoing, and include an overall factor of -1 because an electron line has become a positron line:

$$\left\langle |A|^2 \right\rangle \;=\; g_e^4 \left[\frac{p \cdot q'}{p \cdot q} + \frac{p \cdot q}{p \cdot q'} - 4(\epsilon' \cdot \epsilon)^2 + 2 \right] \;=\; g_e^4 \left[-\frac{\omega'}{\omega} - \frac{\omega}{\omega'} + 4(\epsilon' \cdot \epsilon)^2 - 2 \right] \; . \tag{117}$$

As with Compton scattering, we are using the lab frame in which the electron is at rest so that $p = (mc, \mathbf{0})$ and thus $p \cdot q = m_e \hbar \omega$ and $p \cdot q' = m_e \hbar \omega'$.

Using Eq. (44), the differential cross section in this lab frame is

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{pol.}} \;=\; \frac{1}{16} r_e^2 \, \frac{m_e c}{|\mathbf{p}'|} \, \frac{m_e c^2 (E' + m_e c^2)}{(E' + m_e c^2 - |\mathbf{p}'| \, c \cos\theta)^2} \, \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 4(\epsilon' \cdot \epsilon)^2 + 2 \right] \tag{118}$$

As in the case of Compton scattering, the sum over the photon polarizations is

$$\sum_{i=1}^{2} \sum_{j=1}^{2} (\epsilon_i' \cdot \epsilon_j)^2 \;=\; 1 + \cos^2\theta' \;=\; 2 - \sin^2\theta' \; , \tag{119}$$

where θ' is the angle between the photons' trajectories. Unlike with Compton scattering, θ' is not necessarily the same as the collision angle θ. The other difference from Compton scattering is that now both photons are in the final state and both polarizations are simply summed over, so no factor of $1/2$ is needed to average the polarizations of one of the photons.

Using Eq. (119), the polarization-averaged result is

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{unpol.}} \;=\; \frac{1}{4} r_e^2 \, \frac{m_e c}{|\mathbf{p}'|} \, \frac{m_e c^2 (E' + m_e c^2)}{(E' + m_e c^2 - |\mathbf{p}'| \, c \cos\theta)^2} \, \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \sin^2\theta' \right] \tag{120}$$

For nonrelativistic collision velocities, the calculation may be greatly simplified using $E' \approx m_e c^2 \gg |\mathbf{p}'| \, c$. This is nearly the same as the center-of-mass frame, so the created photons are emitted essentially back-to-back ($\theta' \approx \pi$) and have equal energy $\omega \approx \omega'$. Using these simplifications, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{4} r_e^2 \frac{c}{v'} \; , \tag{121}$$

in which v' is the velocity of the positron relative to the electron.

Simply multiplying by the 4π solid angle, the total cross section is

$$\sigma \;=\; \pi r_e^2 \frac{c}{v'} \; . \tag{122}$$

This answer makes physical sense. The positron must approach the electron within a distance of the order of the classical electron radius to annihilate, and the faster it zips by the less likely it is to interact and annihilate.

2.2.3 Electron-Proton Scattering

Elastic scattering of an electron off a much more massive charged particle such as a proton is called Mott scattering. Provided that the incident electron energy E_1 is less than the proton rest energy, $E_1 << m_p c^2$, one can neglect the internal three-quark structure of the proton and treat it as a point particle. To lowest order, only the Feynman diagram in Fig. 8(a) contributes to this scattering process, and from the Feynman rules its amplitude is

$$A \quad = \quad -\frac{g_e^2}{(p_1 - p_1')^2} \left[\overline{u}(1')\gamma^\mu u(1)\right] \left[\overline{u}(2')\gamma_\mu u(2)\right] \tag{123}$$

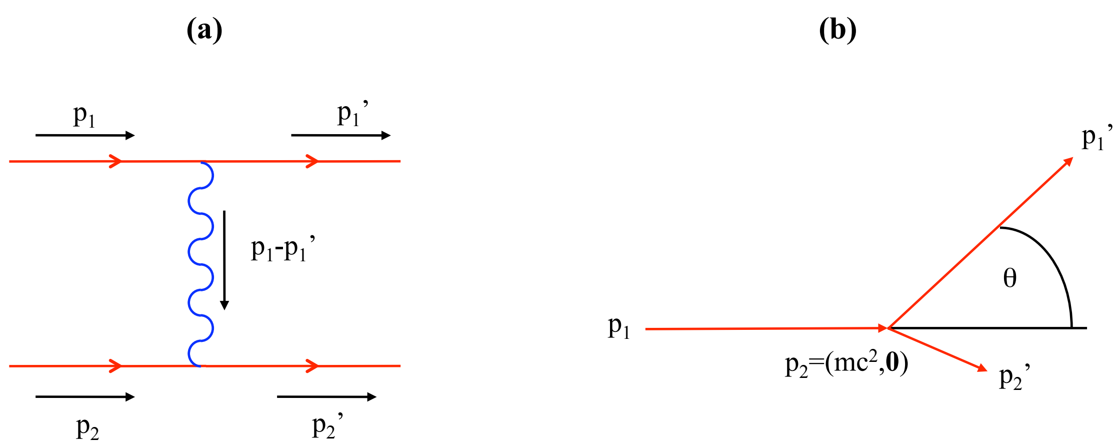

Figure 8. Mott scattering of an electron from a proton (or other very massive charged particle). (a) Only one Feynman diagram contributes to the process in the lowest order. **(b)** Electron scattering angle in the lab frame (proton initially at rest).

One can include a factor of 1/4 to average over the initial spins for the electron and proton:

$$\begin{aligned}
\left\langle |A|^2 \right\rangle &= \frac{1}{4} \sum_{\text{all spins}} \frac{g_e^4}{(p_1 - p_1')^4} \left[\overline{u}(1')\gamma^\mu u(1)\right] \left[\overline{u}(2')\gamma_\mu u(2)\right] \left[\overline{u}(1')\gamma^\nu u(1)\right]^* \left[\overline{u}(2')\gamma_\nu u(2)\right]^* \\
&= \frac{g_e^4}{4(p_1 - p_1')^4} \mathrm{Tr}\left[\gamma^\mu \left(\not{p}_1 + m_e c\right) \gamma^\nu \left(\not{p}_1' + m_e c\right)\right] \mathrm{Tr}\left[\gamma_\mu \left(\not{p}_2 + m_p c\right) \gamma_\nu \left(\not{p}_2' + m_p c\right)\right]
\end{aligned} \tag{124}$$

The first trace in Eq. (124) is

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^\mu \left(\not{p}_1 + m_e c\right) \gamma^\nu \left(\not{p}_1' + m_e c\right)\right] &= \mathrm{Tr}\left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_1'\right] + (mc)^2 \mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right] \\
&= 4\left[p_1^\mu p_1'^\nu + p_1'^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_1')\right] + (m_e c)^2 4 g^{\mu\nu}
\end{aligned} \tag{125}$$

The second trace in Eq. (124) is the same as Eq. (125) except with the substitutions $m_e \to m_p$, $1 \to 2$, and $1' \to 2'$ and with the Greek indices lowered. Thus the amplitude from Eq. (124) becomes

$$\left\langle |A|^2 \right\rangle = \frac{8 g_e^4}{(p_1 - p_1')^4} \left[(p_1 \cdot p_2)(p_1' \cdot p_2') + (p_1 \cdot p_2')(p_2 \cdot p_1') - p_1 \cdot p_1' m_p^2 c^2 - p_2 \cdot p_2' m_e^2 c^2 + 2 m_p^2 m_e^2 c^4\right] \tag{126}$$

Since $m_p \gg m_e$, one may neglect the recoil of the proton. Thus in the lab frame (proton rest frame), the momenta are $p_1 = (E_1/c, \mathbf{p_1})$, $p_1' = (E_1'/c, \mathbf{p_1'})$, and $p_2 = p_2' = (m_p c, \mathbf{0})$, where $E_1 = E_1'$ and $|\mathbf{p_1}| = |\mathbf{p_1'}|$. If θ is the electron scattering angle as shown in Fig. 8(b), the terms in Eq. (126) may be simplified:

$$(p_1 - p_1')^2 \;=\; \frac{E_1^2 + E_1'^2 - 2E_1 E_1'}{c^2} - |\mathbf{p_1}|^2 - |\mathbf{p_1'}|^2 + 2\mathbf{p_1}\cdot\mathbf{p_1'} \;=\; -2\,|\mathbf{p_1}|^2\,(1-\cos\theta) \;=\; -4\,|\mathbf{p_1}|^2 \sin^2\frac{\theta}{2}$$

$$(p_1 \cdot p_1') \;=\; \frac{E_1^2}{c} - \mathbf{p_1}\cdot\mathbf{p_1'} \;=\; m_e^2 c^2 + |\mathbf{p_1}|^2 - |\mathbf{p_1}|^2 \cos\theta \;=\; m_e^2 c^2 + 2\,|\mathbf{p_1}|^2 \sin^2\frac{\theta}{2}$$

$$(p_1 \cdot p_2)(p_1' \cdot p_2') \;=\; (p_1 \cdot p_2')(p_2 \cdot p_1') \;=\; (m_p E_1)^2$$

$$(p_2 \cdot p_2') \;=\; (m_p c)^2$$

$$\Longrightarrow \quad \left\langle |A|^2 \right\rangle \;=\; g_e^4\, \frac{(m_p c)^2 \left[(m_e c)^2 + |\mathbf{p_1}|^2 \cos^2(\theta/2)\right]}{|\mathbf{p_1}|^4 \sin^4(\theta/2)} \tag{127}$$

Compared to the proton, the electron is nearly massless, so the process resembles Compton scattering of a massless particle off a massive one. Therefore, one may plug Eq. (127) into the expression for $d\sigma/d\Omega$ from Eq. (43) with $E_1 = E_1'$:

$$\frac{d\sigma}{d\Omega} \;=\; \left(\frac{\alpha\hbar}{2\,|\mathbf{p_1}|^2 \sin^2(\theta/2)}\right)^2 \left[(m_e c)^2 + |\mathbf{p_1}|^2 \cos^2\frac{\theta}{2}\right] \qquad \textbf{Mott scattering formula} \tag{128}$$

$$= \left(\frac{e^2}{2 m_e v_1^2 \sin^2(\theta/2)}\right)^2 \qquad \text{for } (m_e c)^2 \gg |\mathbf{p_1}|^2 = (m_e v_1)^2 - \textbf{Rutherford scattering formula} \tag{129}$$

2.2.4 Electron-Electron Scattering

Elastic scattering of two identical electrons (or other charged fermions) is called Møller scattering, and is shown in Fig. 9. The amplitude A_a for Fig. 9(a) is the same as Eq. (123) for Fig. 8(a), and the amplitude A_b for Fig. 9(b) is also the same, except with the final particles interchanged, $3 \leftrightarrow 4$. Using QED Feynman Rule 7, the total amplitude A is

$$A \;=\; A_a - A_b \;=\; -\frac{g_e^2\,[\overline{u}(1')\gamma^\mu u(1)]\,[\overline{u}(2')\gamma_\mu u(2)]}{(p_1 - p_1')^2} + \frac{g_e^2\,[\overline{u}(2')\gamma^\mu u(1)]\,[\overline{u}(1')\gamma_\mu u(2)]}{(p_1 - p_2')^2} \tag{130}$$

The spin-averaged square of the amplitude may be written as

$$\left\langle |A|^2 \right\rangle \;=\; \left\langle |A_a|^2 \right\rangle - \langle A_a A_b^* \rangle - \langle A_a^* A_b \rangle + \left\langle |A_b|^2 \right\rangle \tag{131}$$

Since the electrons are on their mass shell, $p_1^2 = p_1'^2 = p_2^2 = p_2'^2 = m_e c^2$, conservation of momentum and energy yields the useful relations

$$p_1 + p_2 = p_1' + p_2' \;\;\rightarrow\;\; p_1^2 + p_2^2 + 2p_1\cdot p_2 = p_1'^2 + p_2'^2 + 2p_1'\cdot p_2' \;\rightarrow\; p_1\cdot p_2 = p_1'\cdot p_2' \tag{132}$$

$$p_1 - p_1' = p_2' - p_2 \;\;\rightarrow\;\; p_1\cdot p_1' = p_2\cdot p_2' \qquad \text{and} \qquad p_1 - p_2' = p_1' - p_2 \;\;\rightarrow\;\; p_1\cdot p_2' = p_2\cdot p_1' \tag{133}$$

For high energies, one can neglect the electron mass when it appears in the equations, $m_e \to 0$. $\left\langle |A_a|^2 \right\rangle$ is the same as Eq. (126) with $m_p \to m_e \to 0$ and may be simplified using Eqs. (132-133):

$$\left\langle |A_a|^2 \right\rangle \;=\; \frac{8g_e^4}{(p_1 - p_1')^4}\,[(p_1\cdot p_2)(p_1'\cdot p_2') + (p_1\cdot p_2')(p_2\cdot p_1')] \;=\; \frac{8g_e^4}{(p_1 - p_1')^4}\left[(p_1\cdot p_2)^2 + (p_1\cdot p_2')^2\right] \tag{134}$$

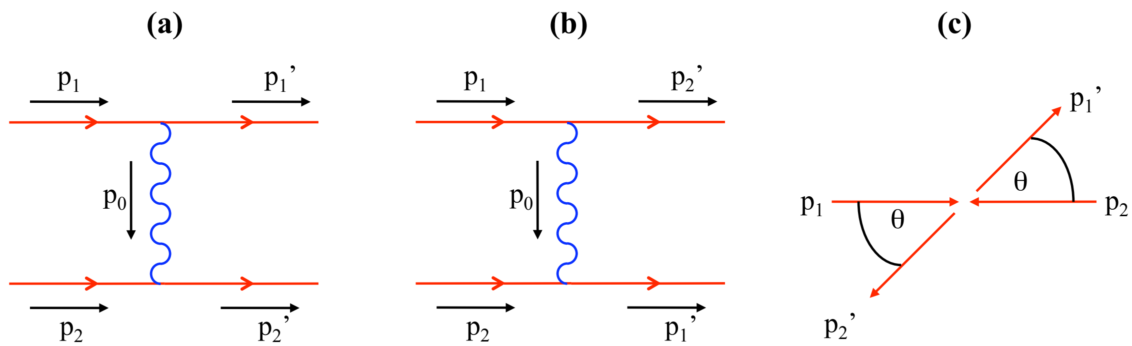

Figure 9. Møller scattering of two electrons. **(a)** and **(b)** The two lowest-order Feynman diagrams contributing to the process are the same except that identities of the outgoing electrons are interchanged. **(c)** The scattering angle in the center-of-mass frame.

By repeatedly using the steps from the derivation of Eq. (80), $\langle A_a A_b^* \rangle$ may be rewritten as a trace

$$\begin{aligned}
\langle A_a A_b^* \rangle &= \frac{1}{4} \sum_{\text{all spins}} \frac{g_e^4 \, [\overline{u}(1')\gamma^\mu u(1)] \, [\overline{u}(2')\gamma_\mu u(2)] \, [\overline{u}(2')\gamma^\nu u(1)]^* \, [\overline{u}(1')\gamma_\nu u(2)]^*}{(p_1 - p_1')^2 (p_1 - p_2')^2} \\
&= \frac{g_e^4}{4(p_1 - p_1')^2 (p_1 - p_2')^2} \mathrm{Tr} \left[\gamma^\mu (\not{p}_1 + m_e c) \gamma^\nu (\not{p}_2' + m_e c) \gamma_\mu (\not{p}_2 + m_e c) \gamma_\nu (\not{p}_1' + m_e c) \right] \\
&= \frac{g_e^4}{4(p_1 - p_1')^2 (p_1 - p_2')^2} \mathrm{Tr} \left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2' \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_1' \right] \qquad \text{since } m_e = 0 \text{ here} \\
&= -8 g_e^4 \frac{(p_1 \cdot p_2)^2}{(p_1 - p_1')^2 (p_1 - p_2')^2} \qquad \text{using Eqs. (94) and (132)} \qquad (135)
\end{aligned}$$

In the CM frame [Fig. 9(c)] with each electron having energy E, one may use Eqs. (48-49) and

$$p_1 \cdot p_2 = \frac{2E^2}{c^2} \qquad p_1 \cdot p_1' = \frac{E^2}{c^2}(1 - \cos\theta) \qquad p_1 \cdot p_2' = \frac{E^2}{c^2}(1 + \cos\theta) \qquad (136)$$

With the substitution $1' \to 2'$, $\langle A_a^* A_b \rangle$ is the same as Eq. (135) and $\left\langle |A_b|^2 \right\rangle$ is the same as Eq. (134), so the total $\left\langle |A|^2 \right\rangle$ is

$$\begin{aligned}
\left\langle |A|^2 \right\rangle &= \frac{8 g_e^4 \{ (p_1 - p_2')^4 [(p_1 \cdot p_2)^2 + (p_1 \cdot p_2')^2] + (p_1 - p_1')^4 [(p_1 \cdot p_2)^2 + (p_1 \cdot p_1')^2] + 2(p_1 \cdot p_2)^2 (p_1 - p_1')^2 (p_1 - p_2')^2 \}}{(p_1 - p_1')^4 (p_1 - p_2')^4} \\
&= 4 g_e^4 \left(1 - \frac{4}{\sin^2\theta} \right)^2 \qquad (137)
\end{aligned}$$

Using Eq. (51), the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{8} \left(\frac{m_e c^2}{E} \right)^2 r_e^2 \left(1 - \frac{4}{\sin^2\theta} \right)^2 \qquad (138)$$

Møller scattering formula
for $E \gg m_e c^2$ in CM frame

2.2.5 Renormalization

As outlined so far, field theory expands a physical process into a series of contributing Feynman diagrams with increasing numbers of vertices. Because the vertex coupling constant is $g \ll 1$, the diagrams with the fewest vertices should yield a good approximation of the physical process. Diagrams of higher order (more vertices) should contribute much less. Unfortunately, if those higher-order diagrams contain closed loops, their contribution is not very small–it is actually infinite.

Renormalization is a technique for (1) convincing ourselves that these infinities are merely mathematical artifacts, not real physical effects, and hiding them by "renormalizing" what we mean by a particle's mass and charge, and (2) correctly calculating the finite (indeed very small) physically observable effects that do arise from loop diagrams after the infinities have been dispatched.

As an example, consider a virtual photon traveling between two vertices in some larger Feynman diagram. Higher-order diagrams represent the possibility that the photon might briefly turn into a virtual electron-positron pair one or more times during transit, as shown in Fig. 10.

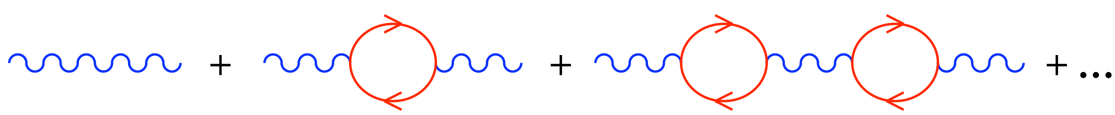

Figure 10. Renormalization of photon lines. An internal (virtual) photon line that is part of some larger Feynman diagram, plus the higher-order processes which can contribute.

Each virtual photon line in these diagrams is calculated with the usual photon propagator,

$$D \equiv \frac{-ig_{\mu\nu}}{k^2} \ . \tag{139}$$

Each electron-positron pair loop (including the vertices on each side of it) introduces a factor of

$$\Pi \equiv g_e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left(\gamma^\mu \frac{i}{\not{p} - mc} \gamma^\nu \frac{i}{\not{p} - \not{k} - mc} \right) \ . \tag{140}$$

This integral is the heart of the problem. Including the d^4p, it has more powers of p in the numerator than the denominator, so when it is integrated up to the infinite momentum that is possible for virtual particles, it blows up and yields an infinite contribution.

The sum of the virtual photon graphs in Fig. 10 (including the vertices at each end) is

$$\begin{aligned} \sum(\text{photon graphs}) &= (ig_e\gamma^\mu)(D + D\Pi D + D\Pi D\Pi D + ...)(ig_e\gamma^\nu) \\ &= -\gamma^\mu g_e^2 D \sum_{n=0}^{\infty} (\Pi D)^n \gamma^\nu \ = \ -\gamma^\mu \left(\frac{g_e^2}{1 - \Pi D} \right) D\gamma^\nu \end{aligned} \tag{141}$$

Our goal is redefine $(g_e/\sqrt{1 - \Pi D})$ as the "observed" coupling constant, so that the resulting expression just looks like a plain virtual photon going between two vertices. That way loop diagrams would always be implicity accounted for without having to explicitly draw them each time.

After several pages of mathematical torture [1-4], the integral for Π can be reduced to

$$\Pi \;=\; ig^{\mu\nu}k^2\,\frac{g_e^2}{12\pi^2}\,\left[\underbrace{\ln\left(\frac{M^2}{m^2}\right)}_{\text{Infinity to hide}} \;-\; \underbrace{f\left(\frac{|k^2|}{m^2c^2}\right)}_{\text{Small finite part to calculate}}\right] , \tag{142}$$

where mass $M \to \infty$ corresponds to an arbitrarily large upper energy limit Mc^2 resulting from the integral and

$$f(x) \;\equiv\; 6\int_0^1 z(1-z)\ln[1+xz(1-z)]dz \;\approx\; \begin{cases} x/5 & \text{for } x \ll 1 \\ \ln x & \text{for } x \gg 1 \end{cases} \tag{143}$$

Plugging Eq. (142) into Eq. (141), one finds

$$\begin{aligned} \sum(\text{photon graphs}) \;&=\; -\gamma^\mu\left(\frac{g_e^2}{1+\frac{g_e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)-\frac{g_e^2}{12\pi^2}f\left(\frac{|k^2|}{m^2c^2}\right)}\right)D\gamma^\nu \\ &=\; -\gamma^\mu\left(\frac{g_{e\text{ renorm}}^2}{1-\frac{g_{e\text{ renorm}}^2}{12\pi^2}f\left(\frac{|k^2|}{m^2c^2}\right)}\right)D\gamma^\nu , \end{aligned} \tag{144}$$

in which the renormalized coupling constant has been defined as

$$g_{e\text{ renorm}} \;\equiv\; \frac{g_e}{\sqrt{1+\frac{g_e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)}} . \tag{145}$$

The renormalized coupling constant $g_{e\text{ renorm}}$ is what is always observed. The unrenormalized coupling constant g_e is not physically measurable and may be defined however we want. In this case, we are defining g_e to contain an infinity that exactly cancels out the infinity from the $\ln(M^2/m^2)$. That sounds strange, but it is legal. Since $g_e \propto e$, renormalizing the coupling constant is equivalent to renormalizing the charge. In general, we will use the measured charge or coupling constant, drop the "renorm" for simplicity, and go back to ignoring loop graphs in internal (virtual) photon lines as we have done until now. For similar reasons, loops in external (real) photon lines can also be ignored as long as the measured charge or coupling constant is used.

Note that after renormalizing away the infinity, a small measurable correction still remains in Eq. (144). The effective coupling constant is a weak function of the momentum k^2 and thus is no longer truly constant; it is referred to as a **running coupling constant**. The physical explanation is that a real electric charge is surrounded by a cloud of virtual electron-positron pairs. The effective charge that is measured depends on how deeply you penetrate through the cloud and approach the bare charge, or equivalently how much momentum you have.

A similar process can be used with loops in electron lines, such as those shown contributing to the virtual electron line in Fig. 11. In this case, the loops represent the electron emitting and then quickly reabsorbing a virtual photon. The propagator for a simple virtual electron line is still

$$S \equiv \frac{i}{\not{p}-mc} , \tag{146}$$

and each loop contributes a factor of

$$\varepsilon \equiv -g_e^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\left(\frac{-ig_{\mu\nu}}{k^2}\right)\left(\frac{i}{\not{p}-\not{k}-mc}\right)\gamma^\nu . \tag{147}$$

As before, simply counting powers of k in this integral (including d^4k) shows that it is infinite.

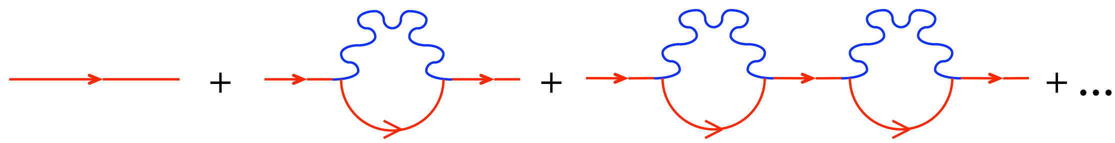

Figure 11. Renormalization of electron lines. An internal (virtual) electron line that is part of some larger Feynman diagram, plus the higher-order processes which can contribute.

The sum of the virtual electron graphs shown in Fig. 11 is

$$\begin{aligned}\sum(\text{electron graphs}) &= S + S\varepsilon S + S\varepsilon S\varepsilon S + ... \\ &= S\sum_{n=0}^{\infty}(\varepsilon S)^n \;=\; \frac{S}{1-\varepsilon S} \\ &= \frac{i}{\not{p} - mc - i\varepsilon} \;=\; \frac{i}{\not{p} - m_{\text{renorm}}c}\,, \end{aligned} \tag{148}$$

where $m_{\text{renorm}} \equiv m + i\varepsilon/c$ is the renormalized electron mass. The bare electron mass that is never observed, m, can be defined to contain an infinity that will exactly cancel the infinity from ε, resulting in a renormalized mass of the actual observed value. Therefore we can use the measured mass value, drop the "renorm," and return to ignoring loop graphs in all electron lines.

By Taylor-expanding the integral in Eq. (147), one can show that it should renormalize charge as well as mass. Fortunately, this contribution to charge renormalization is exactly cancelled out by the contribution from loops at vertices that will be considered in the next section. This miracle is called the Ward Identity. Thus the only net effect on charge renormalization comes from Eq. (145). For more details about renormalization, see [2-4].

2.2.6 Anomolous Magnetic Moment of the Electron

Having considered renormalization due to loops in electron and photon lines, we turn finally to renormalization from loops at vertices. As usual, this leads to an ignorable infinity and a physically measurable small correction, which in this case is the anomolous magnetic moment of the electron.

Before tackling the anomolous moment, we will calculate the basic magnetic moment $\mu_{\mathbf{e}}$ of the electron. This comes from a simple interaction vertex between an electron and an electromagnetic field, as shown in Fig. 12(a). As revealed in Eq. (17), the amplitude of a simple vertex is proportional to the interaction Lagrangian $\mathcal{L}_{\text{Int}}$. We will write this particular calculation in terms of $\mathcal{L}_{\text{Int}}$ instead of the amplitude. Using sneaky tricks, we can rewrite the vertex interaction until it clearly looks like the Lagrangian (negative of the potential energy) of a magnetic moment $\mu_{\mathbf{e}}$ in a magnetic field:

$$\begin{aligned}
\mathcal{L}_{\text{Int}} &= -e\overline{u}(p')\gamma^\mu u(p)A_\mu(q=p'-p) \\
&= -\frac{e}{2m_e c}\overline{u}(p')[\underbrace{(p'+p)^\mu}_{\text{Neglect for nonrelativistic electron \& static magnetic field}} + i\sigma^{\mu\nu}q_\nu]u(p)A_\mu(q) \qquad \text{via Gordon identity (see below)} \\
&= -\frac{e}{2m_e c}\underbrace{\overline{u}(p')\sigma^{\mu\nu}u(p)}_{\text{electron spin } \mathbf{S}}\underbrace{iq_\nu A_\mu(q)}_{\hbar\nabla\times\mathbf{A}=\hbar\mathbf{B}} \\
&= \mu_{\mathbf{e}}\cdot\mathbf{B} \qquad \text{where } \mu_{\mathbf{e}} = -\frac{e\hbar}{2m_e c}\mathbf{S}\,.
\end{aligned} \tag{149}$$

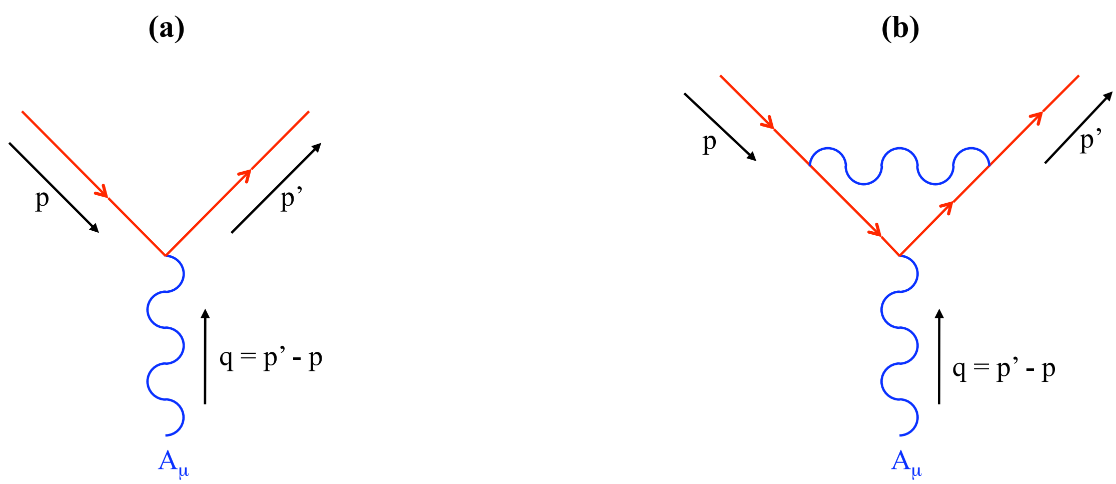

Figure 12. Electron magnetic moment and renormalization of vertices. (a) A simple QED vertex representing an electron interacting with an electromagnetic field may be used to calculate the basic magnetic moment of the electron. **(b)** Loops at the vertex produce unphysical infinities, which may be swept under the rug by the mathematical process of renormalization, and a small observable correction factor, the anomolous magnetic moment of the electron.

The derivation of Eq. (149) used the Gordon identity, which may be proved as follows:

$$\begin{aligned}
\overline{u}(p')\gamma^\mu u(p) &= \frac{1}{2m_e c}\overline{u}(p')[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu]u(p) \qquad \textbf{Gordon identity} \quad (150) \\
\Longrightarrow 0 &= \frac{1}{2m_e c}\overline{u}(p')[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu]u(p) \; - \; \overline{u}(p')\gamma^\mu u(p) \\
&= \overline{u}(p')[p'^\mu + p^\mu + i\sigma^{\mu\nu}p'_\nu - i\sigma^{\nu\mu}p_\nu - 2m_e c\gamma^\mu]u(p) \\
&= \overline{u}(p')[p'^\mu + p^\mu + (\gamma^\nu\gamma^\mu - g^{\mu\nu})p'_\nu - (g^{\nu\mu} - \gamma^\mu\gamma^\nu)p_\nu - 2m_e c\gamma^\mu]u(p) \\
&= \underbrace{\overline{u}(p')(\not{p}' - m_e c)}_{=0 \text{ by Dirac eq.}}\gamma^\mu u(p) + \overline{u}(p')\gamma^\mu \underbrace{(\not{p} - m_e c)u(p)}_{=0 \text{ by Dirac eq.}}
\end{aligned}$$

The loop diagram shown in Fig. 12(b) modifies this process with a higher-order correction term,

$$A = ie\overline{u}(p')(\gamma^\mu + \Lambda^\mu)u(p)A_\mu(q = p' - p) \; , \tag{151}$$

where the correction term is given by the integral

$$\Lambda^\mu = -g_e^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\lambda \left(\frac{-ig_{\lambda\sigma}}{k^2}\right)\gamma^\sigma \left(\frac{i}{\not{p}' - \not{k} - m_e c}\right)\gamma^\mu \left(\frac{i}{\not{p} - \not{k} - m_e c}\right) \; . \tag{152}$$

This integral is logarithmically divergent (with as many powers of k in the denominator as the numerator). After pages of horrifying integration [2-4], Λ^μ may be reduced to the form

$$\Lambda^\mu = \infty\gamma^\mu + \frac{\alpha}{2\pi}\frac{i\sigma^{\mu\nu}(p'-p)^\nu}{2m_e c} \; , \tag{153}$$

where $\alpha \equiv e^2/(\hbar c) \approx 1/137$ is the fine structure constant.

The physically meaningless infinity could be hidden by further renormalizing the charge at the vertex, but we don't even have to do that; it exactly cancels the charge renormalization contribution from Eq. (148) and can be ignored. The second term in Eq. (153) is the important one–a small but measurable correction to the electron magnetic moment, so that the total moment is

$$\mu_{\mathbf{e}} = -\frac{e\hbar}{2m_e c}\left(1 + \frac{\alpha}{2\pi}\right)\mathbf{S} \tag{154}$$

Vertex graphs with more loops would just produce additional ignorable infinities and correction factors that are a factor of α smaller for each extra loop, so they won't be considered here. As it is, Eq. (154) agrees with experiment to within 10^{-6}. While tree graph effects can also be calculated using more classical (non-QED) approaches, only QED can predict loop effects, and as shown by the anomolous magnetic moment of the electron, it gets them right with impressive accuracy.

3 Weak Nuclear Force: Vector Boson and Electroweak Theories

The weak nuclear force is mediated by massive spin-1 bosons, the charged W^+ and W^- and uncharged Z^0, and causes processes such as the decay of muons and neutrons. The weak nuclear force can be combined with electromagnetism into the unified electroweak theory.

3.1 Intermediate Vector Boson Theory

A simple approach to the weak nuclear force is to focus on the $W^\pm$ and ignore the Z^0 and electroweak interactions for now. This is called the Intermediate Vector Boson (IVB) theory, since the W boson is represented by a vector field (spin-1) and acts as an intermediary in weak interactions.

The Lagrangian density for a field of free spin-1 particles of mass m is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi} \left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu \; , \tag{155}$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ as in the case of the photon.

Plugging this Lagrangian into the Euler-Lagrange equation gives the Proca field equation:

$$\partial_\mu F^{\mu\nu} \; + \; \left(\frac{mc}{\hbar}\right)^2 A^\nu \; = \; 0 \qquad \text{Proca eq. for massive spin-1 particles} \tag{156}$$

Note that the Proca equation reduces to the equation for a free photon if $m = 0$.

The momentum-space version of the Proca equation is found by substituting $p_\mu = i\hbar\partial_\mu$:

$$\begin{aligned} \hbar^2 \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \; + \; m^2 c^2 A^\nu \;&=\; 0 \qquad \text{Regular-space Proca equation} \\ \left[\left(-p^2 + m^2 c^2\right) g_{\mu\nu} + p_\mu p_\nu\right] A^\nu \;&=\; 0 \qquad \text{Momentum-space Proca equation} \end{aligned} \tag{157}$$

Using Eq. (27), the propagator for the W in the Feynman rules is i times the (tensor) inverse of the expression in brackets in Eq. (157):

$$\frac{-i}{p^2 - m_W^2 c^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2 c^2}\right) \qquad \text{Propagator for } W \tag{158}$$

The $W^\pm$ mass $m_W = 81.8$ GeV$/c^2$ is so large that $p^2 \ll m_W^2 c^2$ is usually a very good approximation (except for very high-energy collisions):

$$\frac{i g_{\mu\nu}}{m_W^2 c^2} \qquad W \text{ propagator for } p^2 \ll m_W^2 c^2 \tag{159}$$

All leptons and quarks can interact with the $W^\pm$. Figure 13 shows vertices for interactions with the W^-. The W^+ is just the antiparticle of the W^-, so these vertices may be viewed as emission of a W^- or absorption of a W^+. These vertices could also be drawn with all the arrows reversed. For the Feynman rules, the factor for each vertex is

$$\frac{-i g_W}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) \; , \tag{160}$$

in which g_W is the weak coupling constant. Not including the $2\sqrt{2}$ in g_W is simply a convention.

For quarks, this vertex factor needs to be modified slightly. Whereas the weak interaction turns a simple electron into an electron neutrino [Fig. 13(a)], it turns a d' quark state, a superposition of a d and an s quark, into a u quark [Fig. 13(b)]. Likewise, it can turn an s' quark state, a superposition of an s and a d quark, into a c quark [Fig. 13(c)]. As shown in Fig. 14(a), two orthogonal axes may be used to plot the d and s quark states. The d' and s' superpositions are quark states that are obtained by rotating the axes by the Cabibbo angle θ_C, which can be done with a simple rotation matrix such as was used in classical mechanics:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \; . \tag{161}$$

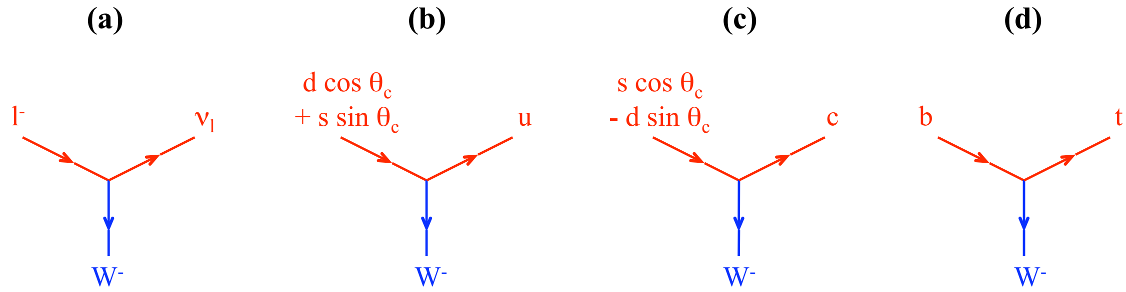

Figure 13. Weak interaction vertices in IVB theory. (a) A lepton l^- (an electron, muon, or tau) can emit a W^- and turn into a neutrino of the corresponding type (ν_e, ν_μ, or ν_τ). **(b)** An initial quark that is a mixture of down and strange quark states [see Fig. 14(a)] can emit a W^- and turn into an up quark. If the type of initial quark is known (d or s), the corresponding factor ($\cos\theta_c$ or $\sin\theta_c$) can simply be incorporated into the vertex factor. (c) Similarly, a strange or down quark can emit a W^- and turn into a charm quark. (d) A bottom quark can emit a W^- and turn into a top quark; in this case there is very little mixing with the other quark types. One can reverse all of the arrows in these vertices, and as always, a particle going one way is equivalent to an antiparticle going the other way.

Note that if $\theta_C = 0$, d' and s' are just the ordinary down and strange quark states, respectively. Because $\theta_C = 13.1^o$ experimentally, there is a small but appreciable amount of mixing between the down and strange quark states in weak interactions. If the quark types in an interaction are known, the appropriate factor of $\cos\theta_C$ or $\sin\theta_C$ may simply be incorporated into the vertex factor.

As shown in Fig. 13(d), the weak interaction generally just turns a bottom quark into a top quark. There is a small amount of mixing with the other quark states, but it is less than 0.1% and the particles involved are so massive that they are seldom created; therefore this mixing will not be considered here. In order to include it, one could generalize Eq. (161) to a 3x3 rotation matrix that relates ordinary d, s, and b quark states to rotated d', s', and b' states. Such a rotation matrix is called the Kobayashi-Maskawa matrix. Incidentally, it is solely a convention to rotate the d, s, and b instead of the corresponding u, c, and t quarks; either choice leads to the same final result.

To look ahead, Figure 14(b) shows that in the electroweak unified theory, a similar rotation of states turns the B and W^3 bosons into the photon and the Z^0 boson, as will be discussed in Section 3.3.

The electron, muon, and tau neutrinos are really only distinguishable by whether they interact with the electron, muon, or tau at a $W^\pm$ vertex. Therefore, any rotation of neutrino states is already built into the definitions of the neutrino types.

Other than the W propagator and the vertex factors noted above, the Feynman rules for the IVB theory of weak interactions are exactly like the Feynman rules for QED.

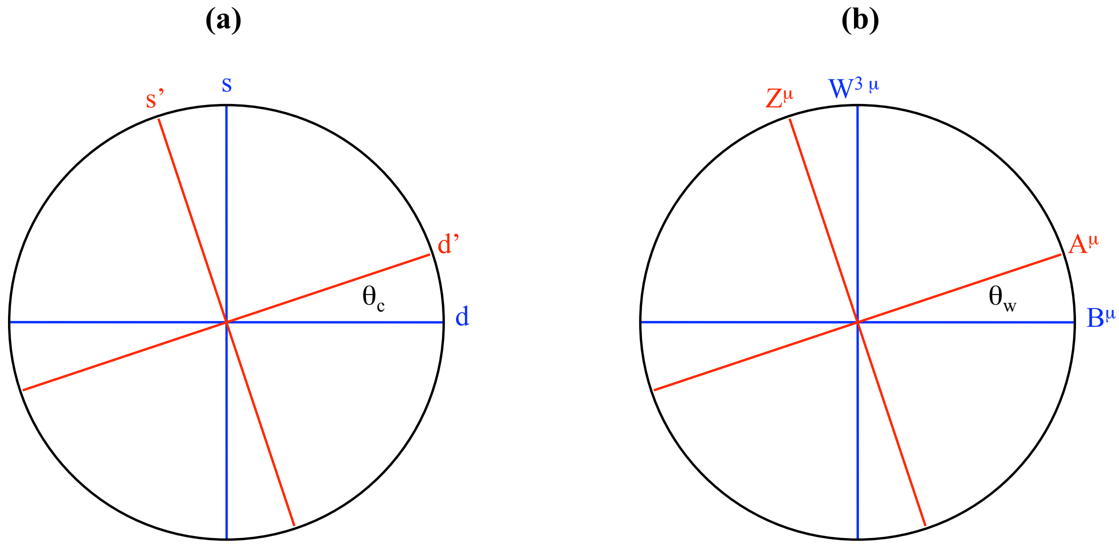

Figure 14. Mixing or rotation of particle states in weak interactions. Two distinct types of particles may be represented by orthogonal axes. Superpositions of those two types of particles may form two new types of particles that are distinct from each other (still on orthogonal axes). This is equivalent to rotating the axes that define the two basic types of particles. **(a)** In interactions with the W^-, down and strange quark states mix to form new orthogonal d' and s' quark states that have been rotated by the Cabibbo angle, where $\theta_c = 13.1^o$ experimentally. **(b)** In the GWS electroweak theory, two "original" neutral spin-1 bosons W^3 and B have mixed to form the two observed neutral spin-1 bosons, Z^0 (represented by the field Z^μ) and the photon (represented by the field A^μ). This mixing is equivalent to rotation by the weak mixing angle θ_w, where $\theta_w = 28.7^o$ experimentally.

3.2 Examples

The weak force can be involved in scattering, but its most important and common effects are certain decay processes. Unfortunately, calculations of weak decays are particularly nasty because they involve three final particles, not two as in Eq. (34). Just take a deep breath and we'll dive in.

3.2.1 Muon Decay

As mentioned earlier, a muon is essentially an overweight electron. It decays into a real electron (plus a muon neutrino and electron anti-neutrino) with a lifetime that will now be calculated. Applying the Feynman rules to the muon decay shown in Fig. 15(a) yields the amplitude,

$$A = \frac{g_W^2}{8m_W^2 c^2}\left[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right]\left[\overline{u}(4)\gamma_\mu(1-\gamma^5)v(2)\right] \ . \tag{162}$$

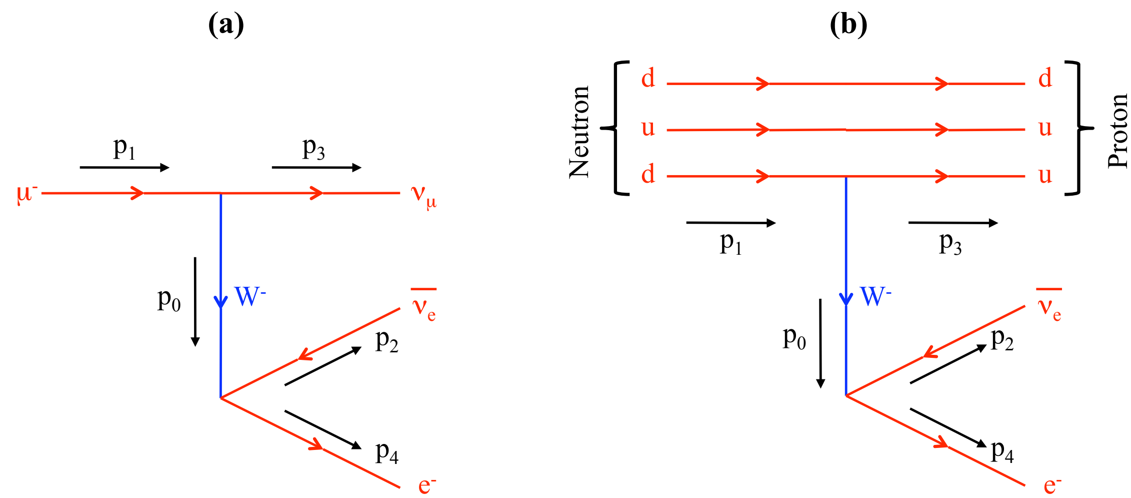

Figure 15. The weak interaction mediates muon and neutron decays. (a) A muon decays into an electron, electron anti-neutrino, and muon neutrino. **(b)** A neutron decays into a proton, electron, and electron anti-neutrino. The neutron and proton are both composed of three quarks; the weak interaction changes one of them from a down to an up quark, thus turning the neutron into a proton.

The spin-averaged and -summed square of the amplitude may be found by using Casimir's trick from Eq. (80) and including a factor of 1/2, since neutrinos have one spin state instead of two:

$$\left\langle |A|^2 \right\rangle = \sum_{\text{spins}} |A|^2 = \frac{1}{2}\frac{1}{64}\left(\frac{g_W}{m_W c}\right)^4 \tag{163}$$

$$\times \underbrace{\mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e c)\gamma^\nu(1-\gamma^5)\ \not{p}_3\right]}_{8\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma}p_{1_\lambda}p_{3_\sigma}\right]} \underbrace{\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\ \not{p}_2\gamma_\nu(1-\gamma^5)(\not{p}_4 + m_\mu c)\right]}_{8\left[p_{2_\mu}p_{4_\nu} + p_{2_\nu}p_{4_\mu} - g_{\mu\nu}(p_2\cdot p_4) - i\epsilon_{\mu\nu\kappa\tau}p_2^\kappa p_4^\tau\right]}$$

from trace theorems:

$$= 2\left(\frac{g_W}{m_W c}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4) \qquad [\text{using } \epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\kappa\tau} = -2(\delta^\lambda_\kappa\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\kappa)] \tag{164}$$

Using momentum/energy conservation in the muon's rest frame, $(p_1 \cdot p_2)$ and $(p_3 \cdot p_4)$ are

$$\begin{aligned}
p_1 = (m_\mu c, \mathbf{0}) \quad &\rightarrow \quad p_1 \cdot p_2 = m_\mu E_2 \\
p_1 = p_2 + p_3 + p_4 \quad &\rightarrow \quad (p_3 + p_4)^2 = (p_1 - p_2)^2 \\
&\rightarrow \quad p_3 \cdot p_4 = -p_1 \cdot p_2 + \frac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2) \\
&\qquad\qquad = -m_\mu E_2 + \frac{c^2}{2}\left(m_\mu^2 + m_{\nu_e}^2 - m_{\nu_\mu}^2 - m_e^2\right) \qquad (165)
\end{aligned}$$

Thus the amplitude squared is

$$\left\langle |A|^2 \right\rangle = \left(\frac{g_W}{m_W c}\right)^4 m_\mu E_2 \left[\left(m_\mu^2 + m_{\nu_e}^2 - m_{\nu_\mu}^2 - m_e^2\right) c^2 - 2 m_\mu E_2\right] \qquad (166)$$

$$\approx \left(\frac{g_W}{m_W c}\right)^4 m_\mu^2 E_2 (m_\mu c^2 - 2E_2)\ , \qquad (167)$$

where m_e and m_ν were neglected because they are much smaller than m_μ.

The decay rate may be found from Eqs. (30) and (167) with $\delta^4(\sum p) = \delta(\sum E/c)\delta(\sum \mathbf{p})$, $E_1 = m_\mu c^2$, $\mathbf{p_1} = 0$, $E_2 \approx |\mathbf{p_2}|\, c$, $E_3 \approx |\mathbf{p_3}|\, c$, and $E_4 \approx |\mathbf{p_4}|\, c$:

$$\begin{aligned}
\Gamma &= \frac{1}{16(2\pi)^5 \hbar m_\mu} \int d^3\mathbf{p_2} \int d^3\mathbf{p_3} \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle}{|\mathbf{p_2}|\ |\mathbf{p_3}|\ |\mathbf{p_4}|} \delta(m_\mu c - |\mathbf{p_2}| - |\mathbf{p_3}| - |\mathbf{p_4}|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4}) \\
&= \frac{1}{16(2\pi)^5 \hbar m_\mu} \int d^3\mathbf{p_2} \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle \delta(m_\mu c - |\mathbf{p_2}| - |\mathbf{p_3}| - |\mathbf{p_4}|)}{|\mathbf{p_2}|\ |\mathbf{p_3}|\ |\mathbf{p_4}|}\ , \qquad (168)
\end{aligned}$$

where now $|\mathbf{p_3}| = |\mathbf{p_2} + \mathbf{p_4}|$ after the integration over the δ^3 .

One can rewrite the variable of integration $d^3\mathbf{p_2}$ using p_3 and the angle θ between $\mathbf{p_2}$ and $\mathbf{p_4}$:

$$\begin{aligned}
p_3 &= \sqrt{p_2^2 + p_4^2 + 2p_2 p_4 \cos\theta} \rightarrow dp_3 = -\frac{p_2 p_4 \sin\theta d\theta}{p_3} \qquad \text{with } p_2 \text{ and } p_4 \text{ held fixed} \\
\Longrightarrow \quad d^3\mathbf{p_2} &= 2\pi \sin\theta\ d\theta\, |\mathbf{p_2}|^2\, dp_2 = -2\pi\, |\mathbf{p_2}|\, dp_2\ dp_3 \frac{|\mathbf{p_3}|}{|\mathbf{p_4}|} \qquad (169)
\end{aligned}$$

Therefore the decay rate is

$$\begin{aligned}
\Gamma &= \frac{1}{(4\pi)^4 \hbar m_\mu} \int_{p_2 - p_4}^{p_2 + p_4} dp_3 \int dp_2 \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle}{|\mathbf{p_4}|^2} \delta(m_\mu c - p_2 - p_3 - p_4) \\
&= \frac{1}{(4\pi)^4 \hbar m_\mu} \int dp_2 \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle}{|\mathbf{p_4}|^2} \qquad (170)
\end{aligned}$$

if $p_2 - p_4 < m_\mu c - p_2 - p_4 < p_2 + p_4$. This condition means that p_2, p_3, and p_4 must each be less than $m_\mu c/2$. By conservation of energy, the three final particles share the energy $m_\mu c^2$ released by the decay of the muon. Because the final particles are relativistic, the three-momentum of each particle is directly proportional to its share in that energy, $p_i = E_i/c$, so the total momentum available to all three particles is $m_\mu c$. By conservation of momentum in the rest frame of the initial muon, no final particle can receive more than half of that total momentum, since there must always be an equal amount of final momentum going in the opposite direction.

Inserting Eq. (164) for $\left\langle |A|^2 \right\rangle$ into Eq. (170) for Γ, one finds

$$\begin{aligned}
\Gamma &= \left(\frac{g_W}{4\pi m_W}\right)^4 \frac{m_\mu}{\hbar c^2} \int \frac{d^3p_4}{p_4^2} \int_{\frac{1}{2}m_\mu c - p_4}^{\frac{1}{2}m_\mu c} dp_2 p_2 \left(m_\mu c - 2p_2\right) \\
&= \left(\frac{g_W}{4\pi m_W}\right)^4 \frac{m_\mu}{\hbar c^2} \int d^3p_4 \left(\frac{1}{2}m_\mu c - \frac{2}{3}p_4\right) \\
&= \left(\frac{g_W}{4\pi m_W}\right)^4 \frac{m_\mu}{\hbar c^2} \int_0^{\frac{1}{2}m_\mu c} dp_4 4\pi p_4^2 \left(\frac{1}{2}m_\mu c - \frac{2}{3}p_4\right) \\
&= \left(\frac{g_W}{m_W}\right)^4 \frac{m_\mu^5 c^2}{12(8\pi)^3\hbar}
\end{aligned} \tag{171}$$

Using the values $g_W \approx 0.66$ and $m_W \approx 82$ GeV/c^2, the correct lifetime of the muon is obtained:

$$\tau = \frac{1}{\Gamma} = 2.2 \cdot 10^{-6} \text{ sec} \tag{172}$$

3.2.2 Beta Decay of an Isolated Neutron

An isolated neutron is slightly unstable and will decay into a proton, electron, and electron antineutrino. This is called beta decay since an electron (beta particle in nuclear physics parlance) is emitted. In some cases, β decay can also happen to a neutron in a nucleus, as will be described in the next section. The Feynman diagram for the decay of an isolated neutron is shown in Fig. 15(b). Because this diagram is so similar to that for muon decay, the squared amplitude for neutron decay is simply a modified form of Eqs. (164) and (166) for muon decay:

$$\begin{aligned}
\left\langle |A|^2 \right\rangle &= 2 f_{\text{fudge}} \cos^2\theta_c \left(\frac{g_W}{m_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4) \\
&= f_{\text{fudge}} \cos^2\theta_c \left(\frac{g_W}{m_W c}\right)^4 m_n E_2 \left[\left(m_n^2 - m_p^2 - m_e^2\right) c^2 - 2 m_n E_2\right]
\end{aligned} \tag{173}$$

In contrast to the muon and muon neutrino in Fig. 15(a), the neutron and proton in Fig. 15(b) are composite particles; the W interacts with only one of the quarks in that composite. The $\cos^2\theta_c$ in Eq. (173) comes from that quark-W vertex. We know the momentum p_1 of the neutron and p_2 of the neutron, but we don't know how much of that momentum belongs to the appropriate quark or how much the other quarks perturb the interaction. Therefore we simply use p_1 and p_2 and include a fudge factor f_{fudge} that we expect to be on the order of 1. Unlike with muon decay, m_e cannot be neglected here because it represents a substantial fraction of the released energy.

The decay rate may be found from Eqs. (30) and (173) with $\delta^4(\sum p) = \delta(\sum E/c)\delta(\sum \mathbf{p})$, $E_1 = m_n c^2$, $\mathbf{p_1} = 0$, $E_2 \approx |\mathbf{p_2}|\, c$, $E_3 = c\sqrt{|\mathbf{p_3}|^2 + m_p^2 c^2}$, and $E_4 = c\sqrt{|\mathbf{p_4}|^2 + m_e^2 c^2}$:

$$\begin{aligned}
\Gamma &= \frac{c^3}{16(2\pi)^5 \hbar m_n} \int d^3\mathbf{p_2} \int d^3\mathbf{p_3} \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle}{E_2 E_3 E_4} \delta\left(m_n c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right) \delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4}) \\
&= \frac{c^3}{16(2\pi)^5 \hbar m_n} \int d^3\mathbf{p_2} \int d^3\mathbf{p_4} \frac{\left\langle |A|^2 \right\rangle}{E_2 E_3 E_4} \delta\left(m_n c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right) ,
\end{aligned} \tag{174}$$

where now $|\mathbf{p_3}| = |\mathbf{p_2} + \mathbf{p_4}|$.

One can rewrite the variable of integration $d^3\mathbf{p_2}$ using E_3 and the angle θ between $\mathbf{p_2}$ and $\mathbf{p_4}$:

$$E_3 \;=\; c\sqrt{p_2^2+p_4^2+2p_2p_4\cos\theta+m_p^2c^2} \to dE_3 \;=\; -\frac{cE_2p_4\sin\theta d\theta}{E_3} \qquad \text{with } p_2 \;\&\; p_4 \text{ held fixed}$$

$$\Longrightarrow\; d^3\mathbf{p_2} \;=\; 2\pi\sin\theta\, d\theta\,|\mathbf{p_2}|^2\,dp_2 \;=\; \frac{-2\pi E_2 dE_2 E_3 dE_3}{c^4\,|\mathbf{p_4}|} \tag{175}$$

$$\text{with the constraint}\quad E_{3_-} < E_3 < E_{3_+}\;, \qquad E_{3\pm} \equiv c\sqrt{p_2^2+p_4^2\pm 2p_2p_4+m_p^2c^2} \tag{176}$$

Thus the decay rate is

$$\begin{aligned}\Gamma \;&=\; \frac{1}{(4\pi)^4\hbar c m_n}\int_{E_{3_-}}^{E_{3_+}} dE_3\int dE_2\int d^3\mathbf{p_4}\frac{\left\langle |A|^2\right\rangle}{E_4\,|\mathbf{p_4}|}\delta(m_nc-\frac{E_2}{c}-\frac{E_3}{c}-\frac{E_4}{c})\\ &=\; \frac{1}{(4\pi)^4\hbar c m_n}\int dE_2\int d^3\mathbf{p_4}\frac{\left\langle |A|^2\right\rangle}{E_4\,|\mathbf{p_4}|}\end{aligned} \tag{177}$$

if $E_{3_-} < m_nc - E_2/c - E_4/c < E_{3_+}$. As with muon decay, this condition simply reflects how much of the total energy and momentum each final particle can get without violating conservation of energy and momentum. Using Eq. (176), this condition may be rewritten in terms of E_2:

$$E_{2_-} < E_2 < E_{2_+}\;, \qquad E_{2\pm} \equiv \frac{\frac{1}{2}(m_n^2-m_p^2+m_e^2)c^4 - m_nc^2E_4}{m_nc^2-E_4\mp|\mathbf{p_4}|\,c} \tag{178}$$

Inserting Eq. (173) for $\left\langle |A|^2\right\rangle$ into Eq. (177) for Γ and using Eq. (178), one finds

$$\begin{aligned}\Gamma \;&=\; \frac{f_{\text{fudge}}\cos^2\theta_c}{\hbar c}\left(\frac{g_W}{4\pi m_W c}\right)^4\int\frac{d^3\mathbf{p_4}}{E_4\,|\mathbf{p_4}|}\int_{E_{2_-}}^{E_{2_+}} dE_2\left[(m_n^2-m_p^2-m_e^2)c^2E_2-2m_nE_2^2\right]\\ &=\; \frac{f_{\text{fudge}}\cos^2\theta_c}{\hbar c}\left(\frac{g_W}{4\pi m_W c}\right)^4\int\frac{d^3\mathbf{p_4}}{E_4\,|\mathbf{p_4}|}\left[(m_n^2-m_p^2-m_e^2)\frac{c^2}{2}\left(E_{2_+}^2-E_{2_-}^2\right)-\frac{2m_n}{3}\left(E_{2_+}^3-E_{2_-}^3\right)\right]\\ &\approx\; \frac{f_{\text{fudge}}\cos^2\theta_c}{\hbar c}\left(\frac{g_W}{4\pi m_W c}\right)^4\int\frac{d^3\mathbf{p_4}}{E_4\,|\mathbf{p_4}|}\frac{4}{c^2}E_4\sqrt{E_4^2-m_e^2c^4}[(m_n-m_p)c^2-E_4]^2\end{aligned} \tag{179}$$

Using $d^3\mathbf{p_4} = 4\pi\,|\mathbf{p_4}|\,E_4dE_4/c^2$, one obtains the spectrum of electron energies emitted in β decay,

$$\frac{d\Gamma}{dE_4} \;=\; \frac{f_{\text{fudge}}\cos^2\theta_c}{\pi^3\hbar}\left(\frac{g_W}{2m_W c^2}\right)^4 E_4\sqrt{E_4^2-m_e^2c^4}[(m_n-m_p)c^2-E_4]^2 \tag{180}$$

As would be expected, the electron energy ranges from just the electron's rest energy m_ec^2 to the full energy release $(m_n-m_p)c^2$, with a peak in between those two extremes.

Defining $a \equiv (m_n-m_p)/m_e$ and doing the integral over E_4, the total decay rate is

$$\Gamma \;=\; \frac{f_{\text{fudge}}\cos^2\theta_c}{4\pi^3\hbar}\left(\frac{g_W}{2m_W c^2}\right)^4(m_ec^2)^5\left[\frac{1}{15}(2a^4-9a^2-8)\sqrt{a^2-1}+a\ln(a+\sqrt{a^2-1})\right] \tag{181}$$

Using the usual numbers $g_W \approx 0.66$ and $m_W \approx 82$ GeV$/c^2$, a fudge factor of $f_{\text{fudge}} \approx 1.54$ gives the correct lifetime of an isolated neutron,

$$\tau \;=\; \frac{1}{\Gamma} \;\approx\; 15\ \text{min} \tag{182}$$

Although this decay process is very similar to muon decay, it is many orders of magnitude slower. The reason is that decays generally proceed much more quickly when there is more energy to be released in the process. From Eqs. (171) and (181), weak-interaction-mediated decay rates vary like $\Gamma \sim (\Delta m)^5$, where Δm is the amount of initial particle mass that is converted to energy. For muon decay, this mass converted to energy is almost the entire mass of the muon, $\Delta m \sim m_\mu$, while for neutron decay it is much smaller, $\Delta m \approx m_n - m_p - m_e \sim m_e$.

3.2.3 Beta Decay within a Nucleus

The results from the previous section for beta decay of an isolated neutron may be extended to beta decay of a neutron (or even a proton) within a nucleus. The weak force coupling constant is more complicated than assumed above. It includes a Cabibbo factor $\cos\theta_c \approx 0.974$ and separate coefficients for two types of coupling: vector ($c_V \approx 1$) and axial ($c_A \approx 1.26$). The variation of the decay rate with the emitted electron energy, $d\Gamma/dE_e$, is important because it gives the energy spectrum of the beta particles [Fig 16(e)]. The $d\Gamma/dE_e$ expression for decay of an isolated neutron can be extended to beta decay of a nucleon within a nucleus by including fudge factors, the **Fermi amplitude** A_F to modify vector coupling (beta particle and neutrino emitted with spins antiparallel) and the **Gamow-Teller amplitude** A_{GT} to modify axial coupling (beta and neutrino emitted with spins parallel):

$$\frac{d\Gamma}{dE_e} = \frac{\cos^2\theta_c}{(4\pi)^3\hbar}\left(\frac{g_w}{m_w c^2}\right)^4 \left(c_V^2\,|A_F|^2 + c_A^2\,|A_{GT}|^2\right) F(Z', E_e) E_e \sqrt{E_e^2 - m_e^2 c^4}\left(\Delta E + m_e c^2 - E_e\right)^2 \quad (183)$$

Equation (183) applies to either electrons or positrons emitted in beta decay. It reduces to the expression for isolated neutron decay for $|A_F|^2 = 1$, $|A_{GT}|^2 = 3$, and $\Delta E = (m_n - m_p - m_e)c^2$.

The function $F(Z', E_e)$ in Eq. (183) accounts for the effect of the final nuclear charge Z' on the probability of the emitted electron appearing at the nucleus ($r = 0$):

$$F(Z', E_e) \quad = \quad \left|\frac{\psi_e(Z', r=0)}{\psi_e(Z'=0, r=0)}\right|^2 \;\approx\; \frac{\frac{Z'e^2}{2\epsilon_o\hbar v}}{\left|1 - \exp\left(\mp\frac{Z'e^2}{2\epsilon_o\hbar v}\right)\right|} \quad (184)$$

where the approximate answer is a nonrelativistic result given by other authors; the relativistic result is much nastier [Emilio Segrè, *Nuclei and Particles* (2nd ed., Benjamin Cummings, Reading, MA, 1977); John M. Blatt and Victor F. Weisskopf, *Theoretical Nuclear Physics* (Dover, New York, 1952); Amos deShalit and Herman Feshbach, *Theoretical Nuclear Physics Volume I: Nuclear Structure* (Wiley, New York, 1974)]. The $\mp$ sign is negative for electrons and positive for positrons. In the limit of low energies or emission velocities $v \to 0$, Eq. (184) becomes

$$F(Z', E_e) \quad \approx \quad \begin{cases} \frac{Z'e^2}{2\epsilon_o\hbar v} & \text{for electrons} \\ \frac{Z'e^2}{2\epsilon_o\hbar v}\,\exp\left(-\frac{Z'e^2}{2\epsilon_o\hbar v}\right) & \text{for positrons} \end{cases} \quad (185)$$

The Coulomb field of the nucleus is attractive for escaping electrons and steals energy from them, or equivalently enhances electron emission at low energies, as shown by Eq. (185). In contrast, the field is repulsive for positrons and accelerates them, or suppresses positron emission at low energies. By analogy with alpha decay [*Nuclear Physics* 2.1], Eq. (185) can also be viewed as a Gamow tunneling factor for positrons escaping from the nucleus.

The essence of Eq. (183) integrated over all energies is defined as the **Fermi integral** $f(Z', \Delta E)$:

$$\left(m_e c^2\right)^5 f(Z', \Delta E) \quad \equiv \quad \int_{m_e c^2}^{\Delta E} dE_e\, F(Z', E_e) E_e \sqrt{E_e^2 - m_e^2 c^4}\left(\Delta E + m_e c^2 - E_e\right)^2 \quad (186)$$

The factor $(m_e c^2)^5$ gives the correct units for the energy factors. For $Z' = 0$, the integral can be evaluated analytically, defining $\epsilon \equiv \Delta E / m_e c^2$:

$$f(Z'=0,\Delta E) = \sqrt{\epsilon^2-1}\left(\frac{1}{30}\epsilon^4 - \frac{3}{20}\epsilon^2 - \frac{2}{15}\right) + \frac{1}{4}\epsilon \ln\left(\epsilon + \sqrt{\epsilon^2-1}\right) \tag{187}$$

$$\approx \begin{cases} 0.2155\left(\frac{\Delta E}{m_e c^2} - 1\right)^{7/2} & \text{for } \Delta E - m_e c^2 \ll m_e c^2 \\ \frac{1}{30}\left(\frac{\Delta E}{m_e c^2}\right)^5 & \text{for } \Delta E \gg m_e c^2 \end{cases} \tag{188}$$

Coulomb effects ($Z' \neq 0$) necessitate numerical evaluation of $f(Z', \Delta E)$ [*Nuclear Physics* 2.2]. However, they generally multiply $f(0, \Delta E)$ by $\sim \exp\left(\frac{2\pi Z'}{137}\frac{c}{v}\right)$ for electron emission and $\sim 0.1 - 0.3$ for positron emission with large Z'.

Using the Fermi integral, the total beta decay rate is

$$\Gamma = \frac{\cos^2\theta_c}{(4\pi)^3}\left(\frac{g_w}{m_w c^2}\right)^4 \frac{(m_e c^2)^5}{\hbar} f(Z',\Delta E)\left(c_V^2 |A_F|^2 + c_A^2 |A_{GT}|^2\right) \tag{189}$$

$$= \frac{G_F^2 m_e^5 c^4}{2\pi^3 \hbar^7} f(Z',\Delta E)\left(c_V^2 |A_F|^2 + c_A^2 |A_{GT}|^2\right)\ , \tag{190}$$

where by convention the **Fermi coupling constant** is defined as

$$G_F \equiv \cos^2\theta \left(\frac{g_w}{m_w c^2}\right)^2 \frac{(\hbar c)^3}{4\sqrt{2}} \approx 8.7 \times 10^{-5}\ \ \mathrm{MeV}\cdot\mathrm{fm}^3 \tag{191}$$

The half-life for beta decay is

$$\tau_{1/2} = \frac{\ln 2}{\Gamma} = \frac{\ln 2\ 2\pi^3\hbar^7}{G_F^2 m_e^5 c^4}\frac{1}{f(Z',\Delta E)}\frac{1}{c_V^2 |A_F|^2 + c_A^2 |A_{GT}|^2} \tag{192}$$

The ft value is the half-life adjusted with the $f(Z', \Delta E)$ dependence removed, so it is only a function of the Fermi and Gamow-Teller amplitudes:

$$ft \equiv f(Z',\Delta E)\tau_{1/2} = \frac{\ln 2\ 2\pi^3\hbar^7}{G_F^2 m_e^5 c^4}\frac{1}{c_V^2 |A_F|^2 + 3c_A^2 |A_{GT}|^2} \approx \frac{6140 \text{ sec}}{|A_F|^2 + 1.59\,|A_{GT}|^2} \tag{193}$$

See *Nuclear Physics* 2.2 for more information on applications to specific nuclear decays.

3.3 Glashow-Weinberg-Salam (GWS) Electroweak Unified Theory

Weak and electromagnetic interactions may be unified into a single interaction Hamiltonian density $\mathcal{H}_{\text{Int}} = -\mathcal{L}_{\text{Int}}$. $\mathcal{H}_{\text{Int}}$ is initially expressed in terms of four uncharged, massive, spin-1 force mediators (W^1, W^2, W^3, and B) and the particles to which they couple, and then combinations of these terms are shown to represent the physically observed force mediators (W^-, W^+, Z^0, and photon).

In Eqs. (75) and (77), the photons that mediate the electromagnetic force were shown to couple to an electric current, and that electric current was defined to be an incoming and outgoing electrically charged particle. More generally, a current (not necessarily electrical now) may be defined as consisting of incoming and outgoing particles that couple to some force-mediating particle at a vertex. For example, in Fig. 13(a) the incoming lepton and outgoing neutrino form a weak force current that couples to the W^- weak force particle. We will now consider a theory in which three force mediators W^i (i = 1, 2, or 3) are coupled to corresponding currents j^i with the weak-force coupling constant g_W and a fourth force-mediating particle B is coupled to a different current j^Y with its own coupling constant g'. This may be represented by the interaction Hamiltonian density

$$\begin{aligned}\mathcal{H}_{\text{Int}} &= g_W \sum_{i=1}^{3} (j^i)_\mu (W^i)^\mu \;+\; \frac{g'}{2}(j^Y)_\mu B^\mu \\ &= \underbrace{\frac{g_W}{\sqrt{2}}(j^-)_\mu (W^-)^\mu \;+\; \frac{g_W}{\sqrt{2}}(j^+)_\mu (W^+)^\mu}_{\mathcal{H}_{\text{Int, charged}}} \;+\; \underbrace{g_W (j^3)_\mu (W^3)^\mu \;+\; \frac{g'}{2}(j^Y)_\mu B^\mu}_{\mathcal{H}_{\text{Int, uncharged}}} \qquad (194)\end{aligned}$$

where the complex combinations $(W^\pm)^\mu \equiv (W^1 \mp iW^2)^\mu/\sqrt{2}$ and $j^\pm_\mu \equiv j^1_\mu \pm i j^2_\mu$ are used to express interactions involving the charged $W^\pm$ in the part $\mathcal{H}_{\text{Int, charged}}$. Note that in this section only, all of the Hamiltonians have been implicitly multiplied by $\sqrt{4\pi/(\hbar c)}$ so that we can work in terms of the g coupling constants instead of quantities like charge.

The currents are defined as

$$(j^i)_\mu \;=\; \frac{1}{4}\;(\overline{\nu_e}\;\;\overline{e})\,\gamma_\mu\,(1-\gamma^5)\,\sigma^i \begin{pmatrix} \nu_e \\ e \end{pmatrix} + \text{similar terms for other leptons \& quarks} \qquad (195)$$

The matrices σ^i are the same as the three 2×2 Pauli spin matrices from Eq. (58). Here they mix the ν_e and e states at interaction vertices the same way they mix spin-up and spin-down states in quantum interactions of a spin-$\frac{1}{2}$ particle.

In accordance with the + and − states, one can define

$$\sigma^\pm \equiv \frac{1}{2}\left(\sigma^1 \pm i\sigma^2\right) \qquad \Longrightarrow \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad (196)$$

The charged currents are then

$$(j^\pm)_\mu \;=\; \frac{1}{2}\;(\overline{\nu_e}\;\;\overline{e})\,\gamma_\mu\,(1-\gamma^5)\,\sigma^\pm \begin{pmatrix} \nu_e \\ e \end{pmatrix} + \text{similar terms for other particles} \qquad (197)$$

The charged interaction Hamiltonian may be further rewritten as

$$\mathcal{H}_{\text{Int, charged}} \;=\; \frac{g_W}{2\sqrt{2}}[\underbrace{\overline{\nu_e}\gamma_\mu(1-\gamma^5)e(W^-)^\mu}_{\text{vertex in Fig. 10(a)}} + \underbrace{\overline{e}\gamma_\mu(1-\gamma^5)\nu_e(W^+)^\mu}_{\text{vertex in Fig. 10(a) with all arrows reversed}}] + \begin{array}{c}\text{similar terms for other}\\ \text{leptons and quarks}\end{array} \qquad (198)$$

$\mathcal{H}_{\text{Int, uncharged}}$ represents interactions involving the remaining two uncharged mediators, the B and W^3. As shown in Fig. 14(b), these particle states may be represented as orthogonal axes, and mixing the two particle states by rotating the axes by the weak mixing angle θ_W produces the two particle states that are actually observed in nature, the photon (field A^μ) and a boson called the Z^0. Although the circumstances are different, this basic idea is similar to the earlier rotation of down and strange quark states by the Cabibbo angle. Whereas the $W^\pm$ bosons mediate charged weak interactions, the Z^0 mediates neutral weak interactions, as will be discussed in a moment.

$\mathcal{H}_{\text{Int, uncharged}}$ may be rewritten in terms of the observed photon and Z^0 particle fields:

$$\mathcal{H}_{\text{Int, uncharged}} = \underbrace{\left(g_W \sin\theta_W j^3_\mu + \frac{g'}{2}\cos\theta_W j^Y_\mu\right) A^\mu}_{\mathcal{H}_{\text{Int, EM}}} + \underbrace{\left(g_W \cos\theta_W j^3_\mu - \frac{g'}{2}\sin\theta_W j^Y_\mu\right) Z^\mu}_{\mathcal{H}_{\text{Int, Z}}} \quad (199)$$

The electromagnetic term is simply $\mathcal{H}_{\text{Int, EM}} = g_e j^{EM}_\mu A^\mu$, so defining

$$j^3_\mu + j^Y_\mu/2 \equiv j^{EM}_\mu \quad (200)$$

yields the relation between the coupling constants,

$$g_e = g_W \sin\theta_W = g'\cos\theta_W . \quad (201)$$

If the W^3 particle were observed, it should have the same mass as its cousins, the W^- and W^+. Since Fig. 14(b) shows that

$$(W^3)^\mu = Z^\mu \cos\theta_W + A^\mu \sin\theta_W , \quad (202)$$

the masses of these particles are related by

$$m_W = m_Z \cos\theta_W + m_{\text{photon}} \sin\theta_W = m_Z \cos\theta_W . \quad (203)$$

Using the experimental mass values $m_W = 81.8$ GeV/c^2 and $m_Z = 92.6$ GeV/c^2 in Eq. (203) indicates that $\theta_W \approx 28.7^o$. This same value for θ_W may be obtained by using the coupling constants $g_e \approx 0.3$ and $g_W \approx 0.66$ in Eq. (201).

By using Eqs. (200) and (201), $\mathcal{H}_{\text{Int, Z}}$ may be expressed in terms of a Z_0 coupling constant g_Z,

$$\mathcal{H}_{\text{Int, Z}} = g_Z\left(j^3_\mu - \sin^2\theta_W j^{EM}_\mu\right) Z^\mu \qquad g_Z \equiv \frac{g_e}{\sin\theta_W \cos\theta_W} . \quad (204)$$

Explicitly writing out the currents and defining q_x to be the charge of particle x, $\mathcal{H}_{\text{Int, Z}}$ is

$$\begin{aligned}\mathcal{H}_{\text{Int, Z}} &= g_Z\left[\frac{1}{4}\overline{\nu_e}\gamma_\mu(1-\gamma^5)\nu_e - \frac{1}{4}\bar{e}\gamma_\mu(1-\gamma^5)e - \sin^2\theta_W \frac{q_\nu}{|e|}\overline{\nu_e}\gamma_\mu\nu_e - \sin^2\theta_W \frac{q_e}{|e|}\bar{e}\gamma_\mu e\right] Z^\mu \\ &\qquad + \text{similar terms for other leptons and quarks} \\ &= \sum_{\text{fermions } f} \frac{g_Z}{2}\bar{f}\gamma_\mu(c^f_V - c^f_A\gamma^5) f Z^\mu , \end{aligned} \quad (205)$$

where the vector c^f_V and axial c^f_A couplings for each fermion are defined in Table 4.

f	c^f_v	c^f_A	**Explanation**
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$	$q_\nu = 0$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_w$	$-\frac{1}{2}$	$q_e = -1$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$	Like ν with $q = \frac{2}{3}$.
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$	$-\frac{1}{2}$	Like e^- with $q = -\frac{1}{3}$.

Table 4. Vector c^f_v and axial c^f_A couplings for each fermion f interacting with the Z^0.

Equation (205) describes the interaction vertex shown in Fig. 16(a). The corresponding vertex factor for the Feynman rules is

$$-i\frac{g_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5) \tag{206}$$

The Z^0 has the same propagator as the $W^\pm$, Eq. (158), with the substitution $m_W \to m_Z$. Whereas the $W^\pm$ is important for decays even at low energies, the Z^0 is really only important in some high energy processes such as that shown in Fig. 16(b).

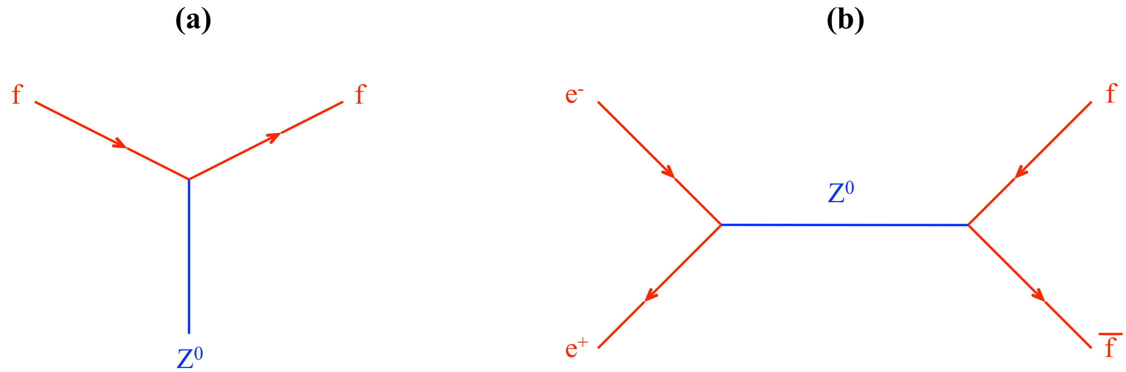

Figure 16. Interactions involving the Z^0. f denotes any fermion. (a) The fundamental interaction vertex. Note that the fermions entering and leaving the vertex are of the same type (unlike with the $W^\pm$ vertex). **(b)** An example of a Z^0 process is the reaction $e^- + e^+ \to$ other fermions. Although this process can be mediated by either the Z^0 or the photon, when the total energy of the colliding particles is $m_Z c^2$, the reaction cross section has a telltale sharp resonance due to the process in (b).

One could say that the electroweak theory is wimpy yet powerful. It combines the forces caused by four different bosons (W^-, W^+, Z^0, and photon) into a unified theory of, well, four different bosons (albeit slightly different ones: W^1, W^2, W^3, and B). That doesn't seem very impressive. On the other hand, it makes many dramatic and experimentally verifiable predictions, such as the ratios m_W/m_Z and g_W/g_Z, as well as the vector and axial couplings for each type of fermion in neutral weak interactions.

In addition to the types of vertices already discussed, electroweak theory contains several three- or four-line vertices that represent interactions just among the $W^\pm$, Z^0, and photon. For example, the photon interacts with charged particles, and the $W^\pm$ are charged. Likewise, the $W^\pm$ and Z^0 can interact with any particles that experience the weak force, including themselves, so there are vertices in which multiple $W^\pm$ and/or Z^0 particles interact. (The photon can also be involved in those vertices if the $W^\pm$ is.) These vertices will not be discussed further here, since they are not very important in most processes and take more time to discuss than they are worth. For more information on them, see [5].

The nonzero masses of the $W^\pm$ and Z^0 keep this electroweak theory from being renormalizable. To circumvent this problem, theoretical physicists with too much time on their hands have invented a new particle called the Higgs boson. The Higgs is an uncharged spin-0 particle so massive that present particle accelerators do not have enough energy to create it. Because it is spin-0, the Higgs boson is itself renormalizable, and through a complicated mechanism called spontaneous symmetry breaking, the Higgs essentially lends mass to the $W^\pm$ and Z^0 bosons. For more information, see [3,5].

4 Strong Nuclear Force: Quantum Chromodynamics

Quantum chromodynamics (QCD) describes the strong nuclear force that binds quarks together to form protons and neutrons, and binds protons and neutrons together to form the nuclei of atoms. Strong force interactions are experienced by particles that have strong force charge, which is called **color** (not related to actual optical colors). The only particles with color are gluons, which actually mediate the strong force, and quarks. Mesons such as pions are combinations of a quark and an antiquark, and baryons such as protons and neutrons are combinations of three quarks. Although QCD describes effects occuring in mesons, baryons, and nuclei, only a few useful QCD calculations of these effects can be made at this point because of mathematical complexities.

4.1 Quantum Chromodynamics (QCD) Theory

The basic QCD interaction vertex is shown in Fig. 17(a). An incoming quark with color #1 (red, blue, or green) emits a gluon and turns into an outgoing quark with color #2. Color is conserved, so the gluon must carry two units of color–one unit of color #1 and one unit of anti-color #2. Although the quark can change colors, it must keep the same flavor (d, u, s, c, b, t). As always, particles going one way may be interpreted as antiparticles going the other way. For instance, the same vertex could represent the *absorption* of a gluon with anti-color #1 and color #2.

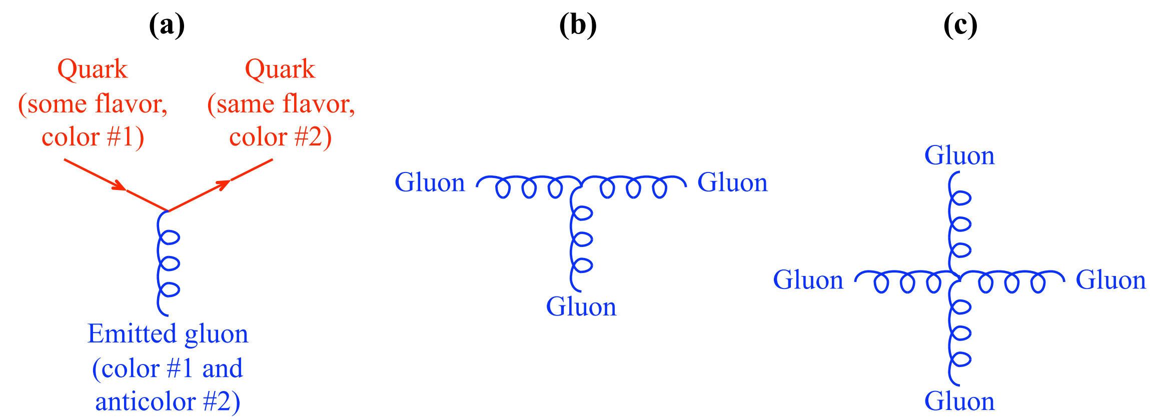

Figure 17. QCD interaction vertices for quarks (straight lines) and gluons (coiled lines). Color can be one of three possible charges, designated red, green, and blue. **(a)** By emitting (or absorbing) a gluon, a quark of color #1 can change to color #2. The gluon must carry the difference in color. Note that the quark flavor (d, u, s, c, b, or t) cannot change during this interaction. **(b)** & **(c)** Since gluons carry color and interact with colored particles, they can interact with themselves in three- and four-gluon vertices.

Quark color is specified by a column matrix c (or row matrix $c^\dagger$, its transposed complex conjugate):

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for red} \qquad c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ for blue} \qquad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for green} \tag{207}$$

Thus for external quark or antiquark lines, one modifies QED rules and writes factors of

$$\begin{array}{llll} u\,c & \text{for incoming quarks} & v\,c & \text{for outgoing antiquarks} \\ \overline{u}\,c^\dagger & \text{for outgoing quarks} & \overline{v}\,c^\dagger & \text{for incoming antiquarks} \end{array} \tag{208}$$

The vertex factor for vertices like Fig. 17(a) is also modified from QED:

$$-\frac{ig_s}{2}\lambda^\alpha\gamma^\mu \ , \qquad \text{where} \quad \lambda^\alpha \equiv \begin{pmatrix} \lambda^\alpha_{rr} & \lambda^\alpha_{rb} & \lambda^\alpha_{rg} \\ \lambda^\alpha_{br} & \lambda^\alpha_{bb} & \lambda^\alpha_{bg} \\ \lambda^\alpha_{gr} & \lambda^\alpha_{gb} & \lambda^\alpha_{gg} \end{pmatrix} \tag{209}$$

Not including the factor of 1/2 in the strong force coupling constant g_s is merely a convention. The Gell-Mann matrix λ^α changes the quark color c from its incoming to its outgoing value in accordance with the gluon type. There are eight Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{210}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

corresponding to the eight different gluons in nature. For example, λ^1 corresponds to a gluon with color state $(r\bar{b} + b\bar{r})/\sqrt{2}$. Note that by convention, a common normalization factor of $1/\sqrt{2}$ has been omitted from the definitions of all the Gell-Mann matrices. It would be simpler if nature had chosen simpler gluon states, such as $r\bar{r}$, $r\bar{b}$, etc., but it perversely chose the above states.

The propagator for virtual quarks or antiquarks of mass m is just the QED fermion propagator. The gluon propagator is the same as the QED photon propagator, but with an added delta function $\delta^{\alpha\beta}$ to ensure that the gluon type is the same at the two vertices connected by the virtual gluon:

$$\frac{i(\not{p} + mc)}{p^2 - m^2c^2} \quad \text{quark propagator} \qquad \qquad \frac{-ig_{\mu\nu}\delta^{\alpha\beta}}{p^2} \quad \text{gluon propagator} \tag{211}$$

The simple Feynman rules presented here are sufficient to calculate effects that will be considered in this section, such as those in Fig. 18(a) and (b). More advanced calculations require additional QCD Feynman rules that will not be described in detail here. Briefly, if external gluon lines existed, they would introduce the same polarization and momentum factor as external photon lines in QED, except the gluon type (1-8) would also need to be noted. Moreover, since gluons have color, they can couple to themselves (unlike photons in QED, which have no electric charge and thus do not couple to themselves). This leads to three- and four-gluon vertices, shown in Fig. 17(b) and (c), and the corresponding vertex factors are mathematically complex and difficult to work with. Likewise, loops in QCD Feynman diagrams require additional calculational techniques called Fadeev-Popov ghosts that will not be discussed here. For more information on advanced QCD rules, see [1,3,6].

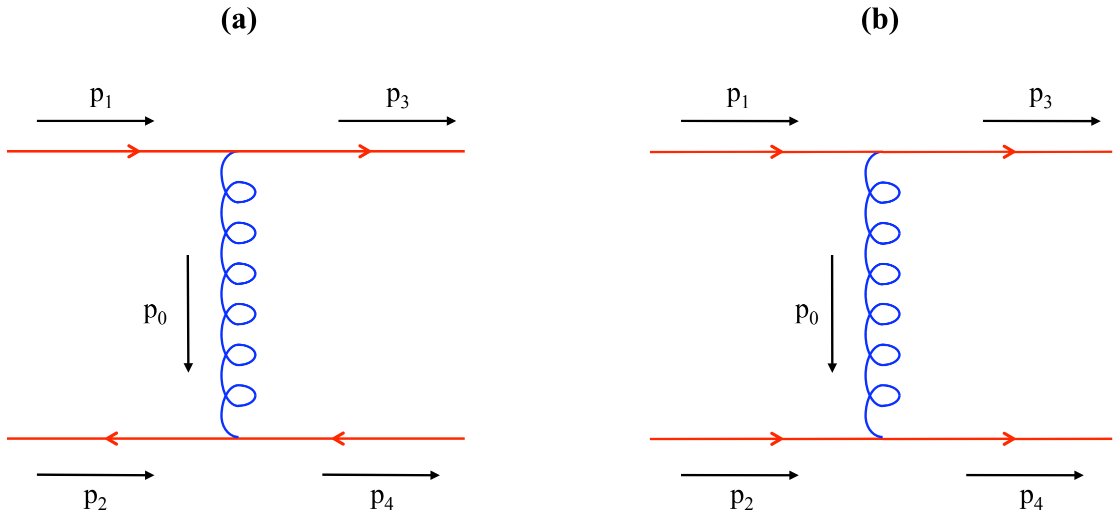

Figure 18. Simplest Feynman diagrams for strong nuclear force between quarks and antiquarks. (a) The strong force between a quark and an antiquark (such as those in a meson) due to the exchange of a single gluon. **(b)** The force between two quarks (such as those in a baryon) due to single-gluon exchange.

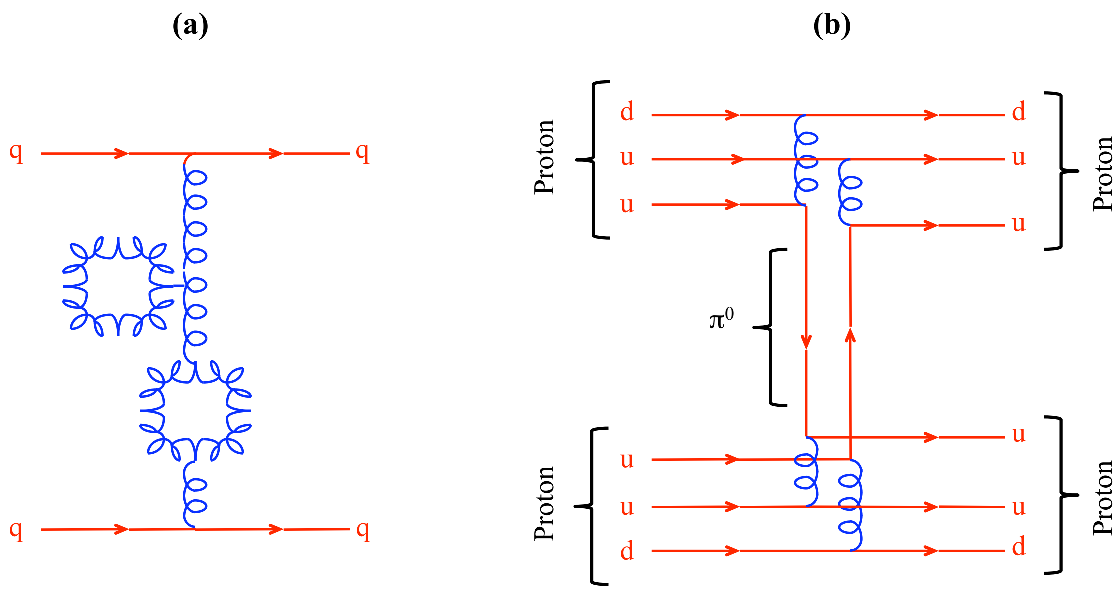

Figure 19. More complicated QCD Feynman diagrams. (a) Gluon loops can appear in diagrams. The net effect of the loops is to increase the effective coupling constant at low energies/long distances and decrease it at high energies/short distances. **(b)** Gluon-mediated interactions between quarks lead to effective meson-mediated interactions between baryons. In this case, two protons exchange a neutral pion. Many similar diagrams contribute to observed strong interactions between baryons.

As in QED, the QCD coupling constant can be renormalized so that the factor g_s in lowest-order Feynman diagrams also implicitly includes the effects of higher order diagrams involving loops of quarks or gluons, such as those shown in Fig. 19(a). This is called a running coupling constant, since it is no longer constant but actually depends on the energy (or equivalently the distance scale) at which the interaction takes place. Two general regimes may be distinguished:

- At high energies (> 300 MeV) or at short distances (<0.7 fm), the renormalized coupling constant is small, $g_s \ll 1$. This means that perturbation theory is valid in this regime. Since the quarks within a baryon or meson are separated by a distance on this scale, they don't interact much with each other, a phenomenon which is called **asymptotic freedom**.
- At lower energies (< 300 MeV) or at longer distances (> 0.7 fm), the renormalized coupling constant increases until it is $g_s > 1$. Therefore, perturbation theory is not valid in this regime, since the series in Eq. (17) does not converge. Because the strong force *increases* with distance, one cannot pull out a free quark; the energy required to remove one quark from a meson or baryon would be sufficient to create new quark partners. The fact that quarks are only found bound up inside mesons and baryons is called **confinement**.

Although free quarks cannot be pulled out of composite particles like baryons, high-energy scattering of electrons off baryons demonstrates that the baryons have an internal structure of three point-like, electrically charged quarks (often called by the older name of partons in this context).

The three color labels have simply been assigned, so they could be arbitrarily interchanged or even redefined as linear combinations of the r, b, and g states. Therefore, an important rule in QCD is that all observed particles must be in colorless (zero net color) states that are invariant under such redefinitions. In fact, as will be shown in Section 4.2, quarks must be in a color-invariant state in order to attract instead of repel each other so that they can form a bound state. This means that mesons and baryons are in the following color-invariant states:

$$\frac{1}{\sqrt{3}}(r\overline{r} + b\overline{b} + g\overline{g}) \qquad \text{meson color state} \tag{212}$$

$$\frac{1}{\sqrt{6}}(rgb + gbr + brg - rbg - grb - bgr) \qquad \text{baryon color state} \tag{213}$$

These states are antisymmetric under the interchange of any two fermions (or symmetric under interchange of a fermion and an antifermion). The only colorless quark combinations are mesons (color-anticolor pairs), baryons (three different colors), anti-baryons (three different anti-colors), and groups of those particles. For example, there are no particles composed of two quarks.

Because mesons and baryons have no net color, gluons do not act between them over physically observable distances. Thus gluons are confined within mesons and baryons just as quarks are.

Since there are three colors of quarks, the Pauli exclusion principle is satisfied even in baryons such as uuu with all of the quark spins in the same direction–each quark is a different color and hence still in a different state.

The masses of u and d can be estimated from the Heisenberg uncertainty principle, $(\Delta p)(\Delta x) \sim \hbar$. If the quarks are confined within a baryon or meson of radius R then $\Delta x \sim R$. Furthermore, if the rest mass of the quarks can be neglected in comparison with the effective mass acquired from their relativistic kinetic energy, then $m_{\text{effective}} \equiv E/c^2 \sim \Delta p/c$. Therefore, one finds

$$m_{\text{effective}} \approx \frac{a\hbar}{Rc} , \tag{214}$$

in which a is a constant on the order of 1. For the measured proton radius $R \approx 1.5 \times 10^{-15}$ m, one obtains the correct quark mass if $a \approx 2.7$:

$$m_u \approx m_d \approx 360 \text{ MeV}/c^2 \text{ in baryons} \tag{215}$$

Mesons have a somewhat larger radius and thus u and d quarks have slightly smaller effective masses in mesons than in baryons.

4.2 Examples

Very few calculations can currently be done with QCD because of complexities associated with the QCD rules, many-body interactions (e.g. 6 quarks and several gluons in 2 interacting nucleons), and the coupling constant becoming too large at longer distances. However, this section will present a few insightful results for mesons and baryons that can be obtained without too much difficulty. In fact, some results arise simply from the requirements on quark wavefunctions, without even involving the QCD Feynman rules for interactions.

4.2.1 Meson Types and Masses

Since mesons are composed of a quark and an antiquark, each of which can be any of the 6 flavors u, d, s, c, b, and t, there are $6 \times 6 = 36$ possible mesons of a given spin. Mesons composed of the three lightest quarks are easiest to create, and Table 4 catalogs these 9 ($= 3 \times 3$) light mesons.

The spin of a meson is 0 if its quark and antiquark spins are antiparallel and 1 if they are parallel. In the ground state, the angular momentum of the quark and antiquark orbiting around each other is zero, just as it is for a ground state (s orbital) electron orbiting in a hydrogen atom. Excited meson states with a nonzero orbital angular momentum that contributes to the total meson spin can occur but will not be considered here.

		Spin 0			Spin 1		
Quarks	Charge	Meson	Mass (MeV/c^2)	Lifetime (sec)	Meson	Mass (MeV/c^2)	Lifetime (sec)
$u\overline{d}$, $d\overline{u}$	± 1	$\pi^{\pm}$	139.6	2.6×10^{-8}	$\rho^{\pm}$	770	4×10^{-24}
$(u\overline{u} - d\overline{d})/\sqrt{2}$	0	π^0	135.0	8.7×10^{-17}	ρ^0	770	4×10^{-24}
$u\overline{s}$, $s\overline{u}$	± 1	$K^{\pm}$	493.7	1.2×10^{-8}	$K^{*\pm}$	892	1×10^{-23}
$d\overline{s}$, $s\overline{d}$	0	K^0, $\overline{K}^{*0}$	497.7		K^{*0}	892	1×10^{-23}
$(u\overline{u} + d\overline{d} - 2s\overline{s})/\sqrt{6}$	0	η	548.8	7×10^{-19}			
$(u\overline{u} + d\overline{d} + s\overline{s})/\sqrt{3}$	0	η'	957.6	3×10^{-21}			
$(u\overline{u} + d\overline{d})/\sqrt{2}$	0				ω	783	7×10^{-23}
$s\overline{s}$	0				ϕ	1020	2×10^{-22}

Table 4. Mesons composed of only the three lightest quarks. Note that linear combinations of $u\overline{u}$, $d\overline{d}$, and $s\overline{s}$ can form three uniquely different particles of a given spin, and that different linear combinations occur for spin-0 and spin-1.

Spin-spin interactions between the quark and antiquark have a large effect on meson masses, which can be accounted for by using the formula

$$m_{\text{meson}} = m_1 + m_2 + A_{\text{meson}} \frac{\mathbf{S_1} \cdot \mathbf{S_2}}{m_1 m_2} \,, \tag{216}$$

where m_1 and m_2 are the masses of the quark and antiquark, $\mathbf{S_1}$ and $\mathbf{S_2}$ are their spins, and A_{meson} is a constant. The product $\mathbf{S_1} \cdot \mathbf{S_2}$ is calculated in the same weird way as it is in the nonrelativistic quantum physics of atomic electron spins:

$$\begin{aligned} J^2 \;=\; (S_1 + S_2)^2 \;=\; S_1^2 + S_2^2 + 2\mathbf{S_1} \cdot \mathbf{S_2} & \\ \Longrightarrow \mathbf{S_1} \cdot \mathbf{S_2} &= \frac{1}{2}\left[J^2 - S_1^2 - S_2^2\right] \\ &= \frac{\hbar^2}{2}\left[j(j+1) - s_1(s_1+1) - s_2(s_2+1)\right] \\ &= \begin{cases} -\frac{3}{4}\hbar^2 & \text{for } j = 0 \\ +\frac{1}{4}\hbar^2 & \text{for } j = 1 \end{cases} \end{aligned} \tag{217}$$

Choosing the magnitude of the spin-spin interactions as $A_{\text{meson}} = 4(m_u/\hbar)^2 160$ MeV/c^2 yields a meson mass formula with a good fit to the experimental mass values in Table 4:

$$m_{\text{meson}} = m_1 + m_2 + 160 \frac{\text{MeV}}{\text{c}^2} \frac{m_u^2}{m_1 m_2} \times \begin{cases} -3 & \text{for } j = 0 \\ +1 & \text{for } j = 1 \end{cases} \tag{218}$$

4.2.2 Meson Binding Potential

The binding potential between a quark and an antiquark in a meson is mediated by the exchange of a gluon, as shown in Fig. 18(a). Using the QCD Feynman rules, the corresponding amplitude is

$$\begin{aligned} A &= i[\overline{u}(3)c_3^\dagger]\left[-i\frac{g_s}{2}\lambda^\alpha\gamma^\mu\right][u(1)c_1]\left[\frac{-ig_{\mu\nu}\delta^{\alpha\beta}}{p_0^2}\right][\overline{v}(2)c_2^\dagger]\left[-i\frac{g_s}{2}\lambda^\beta\gamma^\nu\right][v(4)c_4] \\ &= -\underbrace{\left[\frac{1}{4}\left(c_3^\dagger\lambda^\alpha c_1\right)\left(c_2^\dagger\lambda^\alpha c_4\right)\right]}_{\text{color factor } f=\frac{1}{4}\lambda^\alpha_{c_1c_3}\lambda^\alpha_{c_4c_2}} \frac{g_s^2}{(p_1 - p_3)^2}\left[\overline{u}(3)\gamma^\mu u(1)\right]\left[\overline{v}(2)\gamma_\mu v(4)\right] \end{aligned} \tag{219}$$

The row and column vector c_1 through c_4 simply pick out entries in the rows and columns of the corresponding colors in the λ matrices. Summation over the repeated α index is implied.

The amplitude in Eq. (219) is exactly like that for the QED interaction between opposite charges, except for the additional color factor f and the substitution of the strong coupling constant g_s for the electromagnetic coupling g_e. Therefore, the QCD potential between a quark and an antiquark looks like the QED potential with appropriate modifications:

$$V_{q\overline{q}}(r) = -f\frac{g_s^2\hbar c}{4\pi r} \tag{220}$$

Using the color-invariant state from Eq. (212) for the initial and final meson color states, one finds:

$$\begin{aligned} \langle c_3\overline{c_4}|c_1\overline{c_2}\rangle &= \frac{1}{\sqrt{3}}\left\langle r\overline{r} + b\overline{b} + g\overline{g}\right|\frac{1}{\sqrt{3}}\left|r\overline{r} + b\overline{b} + g\overline{g}\right\rangle \\ &= \frac{1}{3}\left(\langle r\overline{r}|r\overline{r}\rangle + \left\langle r\overline{r}|b\overline{b}\right\rangle + \langle r\overline{r}|g\overline{g}\rangle + \text{ similar terms with } \left\langle b\overline{b}\right| \text{ and } \langle g\overline{g}|\right) \end{aligned} \tag{221}$$

The color factor is the sum of the terms corresponding to Eq. (221):

$$\begin{aligned} f &= \frac{1}{4}\frac{1}{3}\left(\lambda^\alpha_{rr}\lambda^\alpha_{rr} + \lambda^\alpha_{rb}\lambda^\alpha_{br} + \lambda^\alpha_{rg}\lambda^\alpha_{gr} + \text{ similar terms from } \left\langle b\bar{b}\right| + \text{ similar terms from } \langle g\bar{g}|\right) \\ &= \frac{1}{4}\left(\lambda^\alpha_{rr}\lambda^\alpha_{rr} + \lambda^\alpha_{rb}\lambda^\alpha_{br} + \lambda^\alpha_{rg}\lambda^\alpha_{gr}\right) \end{aligned} \tag{222}$$

The factor of 1/3 cancelled the two other sets of similar terms in Eq. (222). For each term, only two λ matrices yield nonzero results, so by looking over Eq. (210), the sums over the λ matrices are

$$f = \frac{1}{4}\left[\left(1\cdot 1 + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) + (1\cdot 1 - i\cdot i) + (1\cdot 1 - i\cdot i)\right] = \frac{4}{3} \tag{223}$$

For comparison, one could consider a meson in a non-color-invariant state like the gluon states in Eq. (210). For example, using the color state corresponding to λ^1, $(r\bar{b} + b\bar{r})\sqrt{2}$, for the initial and final meson color states, one finds:

$$\begin{aligned} \langle c_3\overline{c_4}|c_1\overline{c_2}\rangle &= \frac{1}{\sqrt{2}}\left\langle r\bar{b} + b\bar{r}\right| \frac{1}{\sqrt{2}}\left|r\bar{b} + b\bar{r}\right\rangle \\ &= \frac{1}{2}\left(\left\langle r\bar{b}|r\bar{b}\right\rangle + \left\langle r\bar{b}|b\bar{r}\right\rangle + \left\langle b\bar{r}|r\bar{b}\right\rangle + \langle b\bar{r}|b\bar{r}\rangle\right) \end{aligned} \tag{224}$$

As before, the color factor is the sum of the terms corresponding to Eq. (224):

$$\begin{aligned} f &= \frac{1}{4}\frac{1}{2}\left(\lambda^\alpha_{rr}\lambda^\alpha_{bb} + \lambda^\alpha_{rb}\lambda^\alpha_{rb} + \lambda^\alpha_{br}\lambda^\alpha_{br} + \lambda^\alpha_{bb}\lambda^\alpha_{rr}\right) \\ &= \frac{1}{4}\frac{1}{2}\left[\left(-1\cdot 1 + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) + \left(1^2 + (-i)^2\right) + \left(1^2 + i^2\right) + \left(-1\cdot 1 + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right)\right] = -\frac{1}{6} \end{aligned} \tag{225}$$

The same result would be obtained for any of the other non-color-invariant states. Putting the color factors from Eqs. (223) and (225) into Eq. (220), one finds the quark-antiquark binding potentials in a meson in different color states:

$$V_{q\bar{q}}(r) = -\frac{4}{3}\frac{g_s^2\hbar c}{4\pi r} \quad \text{for color-invariant state} \qquad V_{q\bar{q}}(r) = +\frac{1}{6}\frac{g_s^2\hbar c}{4\pi r} \quad \text{for other states} \tag{226}$$

Thus the potential between the quark and antiquark in a meson is attractive if the particles are in the color-invariant state and repulsive if they are in a non-color-invariant state. This helps to explain why mesons are always in the color-invariant state.

Although the strong force is actually caused by quarks exchanging gluons, confinement of the quarks and gluons within mesons and baryons makes it effectively appear as if the strong force is caused by baryons exchanging mesons. For example, Fig. 19(b) shows two protons exchanging a neutral pion, and this Feynman diagram reveals all of the composite quark and gluon interactions involved in that process. There are actually several Feynman diagrams that could contribute to pion-mediated proton interactions. At closer distances, exchanges of multiple pions or of mesons of larger mass also become important.

The strong nuclear force among protons and neutrons is what holds the nucleus of an atom together despite the electrostatic repulsion among the protons. Because the mesons mediating this strong interaction have mass, the resulting interaction among protons and neutrons is a short-range Yukawa potential like Eq. (9) that can only overcome the electrostatic repulsion among protons at very short distances ($\sim 1 - 2$ fm).

4.2.3 Baryon Types and Masses

Baryons are composed of three quarks, and each quark can be any of the six flavors u, d, s, c, b, and t. Thus there are a large number of possible baryons. In practice, only baryons composed of the three lightest quarks can be easily studied in experiments:

		Spin 1/2			Spin 3/2		
Quarks	Charge	Baryon	Mass (MeV/c^2)	Lifetime (sec)	Baryon	Mass (MeV/c^2)	Lifetime (sec)
uuu	+2	–			Δ^{++}	1232	6×10^{-24}
uud	+1	p	938.3	∞	Δ^{+}	1232	6×10^{-24}
udd	0	n	939.6	900	Δ^{0}	1232	6×10^{-24}
ddd	−1	–			Δ^{-}	1232	6×10^{-24}
uus uds dds	+1 0 −1	Σ^+ Σ^0 Σ^-	1190 1190 1190		Σ^{*+} Σ^{*0} Σ^{*-}	1385 1385 1385	2×10^{-23} 2×10^{-23} 2×10^{-23}
uds	0	Λ	1116				
uss, dss	0, −1	Ξ^0, Ξ^-	1320	2×10^{-10}	Ξ^{*0}, Ξ^{*-}	1533	7×10^{-23}
sss	−1	–			Ω^-	1672	8×10^{-11}

Table 5. Baryons composed of only the three lightest quarks.

Including flavor, spin, color, and space, the wavefunction for the quarks in a baryon must be antisymmetric under interchange of any two quarks. The baryon color state in Eq. (213) is antisymmetric, so the remaining parts of the wavefunction multiplied together must be symmetric. Therefore, baryons with symmetric, identical flavor states (e.g., uuu, ddd, and sss) must have symmetric, identical spin states (all pointing the same way, so total spin must be 3/2 and cannot be 1/2). Because the quarks in uds are all different flavors, multiple wavefunctions with different symmetry properties are possible, and hence there are two uds spin-1/2 baryons, Λ and Σ^0.

Spin-spin interactions between quarks have a large effect on baryon masses as well as meson masses:

$$m_{\text{baryon}} = m_1 + m_2 + m_3 + A_{\text{baryon}}\left(\frac{\mathbf{S_1}\cdot\mathbf{S_2}}{m_1 m_2} + \frac{\mathbf{S_2}\cdot\mathbf{S_3}}{m_2 m_3} + \frac{\mathbf{S_1}\cdot\mathbf{S_3}}{m_1 m_3}\right) . \tag{227}$$

The baryon's three quarks have masses m_1, m_2, and m_3 and spins $\mathbf{S_1}$, $\mathbf{S_2}$, and $\mathbf{S_3}$, and A_{baryon} is a constant. When $m_1 = m_2 = m_3$, the spin-spin interaction calculation may be simplified using

$$\begin{aligned}
J^2 = (S_1+S_2+S_3)^2 &= S_1^2+S_2^2+S_3^2+2(\mathbf{S_1}\cdot\mathbf{S_2}+\mathbf{S_2}\cdot\mathbf{S_3}+\mathbf{S_1}\cdot\mathbf{S_3}) \\
\Longrightarrow \mathbf{S_1}\cdot\mathbf{S_2}+\mathbf{S_2}\cdot\mathbf{S_3}+\mathbf{S_1}\cdot\mathbf{S_3} &= \frac{1}{2}\left[J^2 - S_1^2 - S_2^2 - S_3\right] \\
&= \frac{\hbar^2}{2}\left[j(j+1) - s_1(s_1+1) - s_2(s_2+1) - s_3(s_3+1)\right] \\
&= \begin{cases} -\frac{3}{4}\hbar^2 & \text{for } j=\frac{1}{2} \\ +\frac{3}{4}\hbar^2 & \text{for } j=\frac{3}{2} \end{cases}
\end{aligned} \tag{228}$$

Choosing the magnitude of the spin-spin interactions as $A_{\text{baryon}} = 4(m_u/\hbar)^2 50$ MeV/c^2 yields a good fit to the experimental proton, neutron, and Δ mass values in Table 5:

$$m_{\text{baryon}} = m_1 + m_2 + m_3 + \begin{cases} -150\,\frac{m_u^2}{m_1 m_2}\,\frac{\text{MeV}}{\text{c}^2} & \text{for } j=\frac{1}{2} \\ 150\,\frac{m_u^2}{m_1 m_2}\,\frac{\text{MeV}}{\text{c}^2} & \text{for } j=\frac{3}{2} \end{cases} \tag{229}$$

Masses of baryons whose component quarks have unequal masses can be calculated in similar but more lengthy fashions, as shown in [1].

4.2.4 Baryon Binding Potential

The quarks in a baryon are bound together by the exchange of gluons. The amplitude of the diagram in Fig. 18(b) for gluon exchange between any two of the quarks is

$$\begin{aligned} A &= i[\overline{u}(3)c_3^\dagger]\left[-i\frac{g_s}{2}\lambda^\alpha\gamma^\mu\right][u(1)c_1]\left[\frac{-ig_{\mu\nu}\delta^{\alpha\beta}}{p_0^2}\right][\overline{u}(4)c_4^\dagger]\left[-i\frac{g_s}{2}\lambda^\beta\gamma^\nu\right][u(2)c_2] \\ &= -\underbrace{\left[\frac{1}{4}\left(c_3^\dagger\lambda^\alpha c_1\right)\left(c_4^\dagger\lambda^\alpha c_2\right)\right]}_{\text{color factor } f=\frac{1}{4}\lambda^\alpha_{c_1c_3}\lambda^\alpha_{c_2c_4}}\frac{g_s^2}{(p_1-p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{u}(4)\gamma_\mu u(2)] \end{aligned} \tag{230}$$

This amplitude is the same as that for QED interactions between like charges, except for the additional color factor f and the substitution of g_s for g_e. Thus by analogy, the QCD potential between any two quarks in a baryon is

$$V_{qq}(r) = +f\frac{g_s^2\hbar c}{4\pi r} \tag{231}$$

As with mesons, the binding potential within baryons can be calculated for color-invariant and non-color invariant states. The color-invariant state for quarks in a baryon was given in Eq. (213). Considering only the first two quarks and assuming they begin and end in that state, one finds

$$\begin{aligned} \langle c_3c_4|c_1c_2\rangle &= \frac{1}{\sqrt{6}}\langle rg-rb+gb-gr+br-bg|\frac{1}{\sqrt{6}}|rg-rb+gb-gr+br-bg\rangle \\ &= \frac{1}{6}\left[\left(\langle rg|rg\rangle-\langle rg|rb\rangle+\langle rg|gb\rangle-\langle rg|gr\rangle+\langle rg|br\rangle-\langle rg|bg\rangle\right)+5\text{ more sets of terms}\right] \end{aligned} \tag{232}$$

All 6 sets of terms yield the same results, so one may limit the calculation to the explicitly shown set of terms and multiply by 6. The color factor is the sum of the corresponding terms:

$$f = \frac{1}{4}\left(\lambda^\alpha_{rr}\lambda^\alpha_{gg}-\lambda^\alpha_{rr}\lambda^\alpha_{bg}+\lambda^\alpha_{gr}\lambda^\alpha_{bg}-\lambda^\alpha_{gr}\lambda^\alpha_{rg}+\lambda^\alpha_{br}\lambda^\alpha_{rg}-\lambda^\alpha_{br}\lambda^\alpha_{gg}\right) = -\frac{2}{3} \tag{233}$$

Using $(rb+br)/\sqrt{2}$ as a typical non-color-invariant initial and final quark state, one finds

$$\langle c_3c_4|c_1c_2\rangle = \frac{1}{\sqrt{2}}\langle rb+br|\frac{1}{\sqrt{2}}|rb+br\rangle = \frac{1}{2}\left(\langle rb|rb\rangle+\langle rb|br\rangle+\langle br|rb\rangle+\langle br|br\rangle\right) \tag{234}$$

Thus the color factor for this state is

$$f = \frac{1}{4}\frac{1}{2}\left(\lambda^\alpha_{rr}\lambda^\alpha_{bb}+\lambda^\alpha_{br}\lambda^\alpha_{rb}+\lambda^\alpha_{rb}\lambda^\alpha_{br}+\lambda^\alpha_{bb}\lambda^\alpha_{rr}\right) = +\frac{1}{3} \tag{235}$$

This result holds for other non-color-invariant states too. Using the color factors from Eqs. (233) and (235) in Eq. (231), the binding potential between each pair of quarks in a baryon is

$$V_{qq}(r) = -\frac{2}{3}\frac{g_s^2\hbar c}{4\pi r}\ \text{ for color-invariant state} \qquad V_{qq}(r) = +\frac{1}{3}\frac{g_s^2\hbar c}{4\pi r}\ \text{ for other states} \tag{236}$$

As was the case with mesons, baryons must be in a color-invariant state in order for the potentials between their component quarks to be attractive instead of repulsive.

4.2.5 Baryon Magnetic Moments

Using quark wavefunctions, one can calculate the magnetic moments of baryons. The magnetic moment of a Dirac spin-1/2 point particle of mass m and charge q (ignoring radiative corrections) is $\mu = q\hbar/(2mc)$ [cgs units]. Thus the magnetic moments of u and d quarks are

$$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u c} \qquad \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d c} \tag{237}$$

The magnetic moment of a proton or neutron is simply the sum of the moments of its quarks (subtracting spins of antiparallel quarks), since the quarks have no orbital angular momentum.

The flavor and spin state for the quarks in a spin-up proton is

$$\begin{aligned} |p\uparrow\rangle &= \frac{1}{\sqrt{18}}\left(2\,|u\uparrow\; u\uparrow\; d\downarrow\rangle + 2\,|u\uparrow\; d\downarrow\; u\uparrow\rangle + 2\,|d\downarrow\; u\uparrow\; u\uparrow\rangle - |u\uparrow\; u\downarrow\; d\uparrow\rangle - |u\uparrow\; d\uparrow\; u\downarrow\rangle\right. \\ &\quad \left. - |d\uparrow\; u\uparrow\; u\downarrow\rangle - |u\downarrow\; u\uparrow\; d\uparrow\rangle - |u\downarrow\; d\uparrow\; u\uparrow\rangle - |d\uparrow\; u\downarrow\; u\uparrow\rangle\right) \end{aligned} \tag{238}$$

This state is antisymmetric under simultaneous interchange of flavor and spin between any two quarks. The terms with two identical permutations have coefficients of 2, and the normalization factor $1/\sqrt{18}$ is included since the sum of the squares of each term is 18.

The net number of spin-up u quarks averaged over all of the terms in Eq. (238) is

$$\langle \#u\uparrow - \#u\downarrow\rangle = \sum_{\text{terms}} \left(\begin{array}{c}\text{coefficient with}\\ \text{normalization}\end{array}\right)^2 (\#u\uparrow - \#u\downarrow \text{ in each term}) = \frac{4}{3} \tag{239}$$

Likewise the net number of spin-up d quarks is $\langle \#d\uparrow - \#d\downarrow\rangle = -1/3$.

The magnetic moment of the proton is the sum of the magnetic moments of its constituent u and d quarks, appropriately weighted for the net number of each quark type pointed in the same direction:

$$\begin{aligned} \mu_p &= \langle \#u\uparrow - \#u\downarrow\rangle\,\mu_u + \langle \#d\uparrow - \#d\downarrow\rangle\,\mu_d \\ &= \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = \frac{8}{9}\frac{e\hbar}{2m_u c} + \frac{1}{9}\frac{e\hbar}{2m_d c} \\ &\approx \frac{e\hbar}{2m_u c} \approx 2.79\frac{e\hbar}{2m_p c} \end{aligned} \tag{240}$$

The last two approximations in Eq. (240) were made since $m_u \approx m_d \approx m_p/2.79$.

The neutron magnetic moment is found by simply interchanging $u \leftrightarrow d$ in the above calculation of the proton magnetic moment, since neutrons have ddu and protons have uud quarks:

$$\begin{aligned} \mu_n &= \frac{4}{3}\mu_d - \frac{1}{3}\mu_u = -\frac{4}{9}\frac{e\hbar}{2m_d c} - \frac{2}{9}\frac{e\hbar}{2m_u c} \\ &\approx -\frac{2}{3}\frac{e\hbar}{2m_u c} \approx -1.86\frac{e\hbar}{2m_p c} \end{aligned} \tag{241}$$

The results of Eqs. (240) and (241) are in good agreement with the experimental values of $\mu_p = 2.793\ e\hbar/(2m_p c)$ and $\mu_n = -1.913\ e\hbar/(2m_p c)$, especially considering the uncertainty in the quark masses and the fact that radiative corrections were not included.

5 Gravitational Force: General Relativity and Quantum Gravity

Like the other fundamental forces, gravitation can be described by a classical field (using general relativity) or equivalently by interactions involving a force-mediating quantum particle (the graviton). The basic properties of the graviton can be defined, and it can be shown that Einstein's equation for the gravitational field in general relativity is equivalent to a quantum field theory of interacting gravitons. Unfortunately, a full-fledged quantum theory of gravity has not yet been developed, due to difficulties that will be explained briefly. Nonetheless, a few effects involving both gravitation and quantum theory can be calculated, especially standard quantum effects that are altered if they occur in a classically treated gravitational field.

If gravitational force is mediated by quantum particles, gravitons, one can deduce what the fundamental properties of these particles must be. First of all, gravitons must be massless; otherwise gravitational potentials would look like the exponential Yukawa potential of Eq. (9) instead of the familiar $\sim 1/r$ form. Moreover, gravitons must have spin 2. As discussed in Section 1.1.2, force-mediating particles must have integer spin. Section 1 showed that spin-0 mediators lead to simple scalar field expressions, while Section 2 showed that spin-1 mediators like the photon lead to vector field equations such as electromagnetism. By extension, a spin-2 mediator would lead to tensor field equations, which is exactly what general relativity involves. Higher spin numbers would lead to even more complicated field equations and would not fit the observed behavior of gravity.

As discussed in *Special and General Relativity* 3, Einstein's equation for the gravitational field is

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \;=\; \frac{8\pi G}{c^2} T_{\mu\nu} \qquad \textbf{Einstein's equation} \tag{242}$$

in which the quantities describing gravitational curvature of spacetime are the Einstein tensor $G_{\mu\nu}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar $R \equiv R_{\alpha\beta} g^{\alpha\beta}$. The gravitational fields are caused by the source term $T_{\mu\nu}$, the stress-energy tensor describing any energy (or mass) or momentum that is present.

Einstein's field equation may be derived from the Lagrangian density

$$\mathcal{L} \;=\; \sqrt{\det(g_{\mu\nu})} \left(-\frac{c^4}{16\pi G} + \mathcal{L}_{\text{everything except gravity}} \right) , \tag{243}$$

where $G \approx 6.67 \times 10^{-8}\ \frac{\text{cm}^3}{\text{g sec}^2}$ is Newton's gravitational constant and $\mathcal{L}_{\text{everything except gravity}}$ is the Lagrangian density for any particles or fields (except gravity) that are present.

The spacetime metric $g_{\mu\nu}$ may be separated into a constant background metric (here assumed to be the Minkowski metric $\eta_{\mu\nu}$ of flat spacetime) and a perturbation $f_{\mu\nu}$,

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} + \frac{\sqrt{16\pi\hbar}}{p_p} f_{\mu\nu} , \qquad \text{with} \tag{244}$$

$$p_p \;\equiv\; \sqrt{\frac{\hbar c^3}{G}} \qquad \textbf{Planck momentum} \tag{245}$$

The weird constant in front of $f_{\mu\nu}$ in Eq. (244) ensures that $f_{\mu\nu}$ will have the right dimensions to be a quantum field of massless spin-2 particles while keeping the corresponding perturbation to the metric dimensionless (since the metric is dimensionless). In *Relativity* 3.4, these constants are simply absorbed into the definition of the perturbation to create a dimensionless perturbation of the metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \equiv (\sqrt{16\pi\hbar}/p_p) f_{\mu\nu}$.

The fundamental constants $\hbar$, c, and G may be combined to yield a characteristic momentum p_p (or other quantity) at which quantum gravitational effects become important. For more information on the definitions and physical meaning of this **Planck scale**, see *Relativity* 6.2.

If the perturbation is small,

$$|h_{\mu\nu}| \equiv \left| \frac{\sqrt{16\pi\hbar}}{p_p} f_{\mu\nu} \right| \ll 1 , \tag{246}$$

one obtains linearized field equations as shown in *Relativity* 3.4.

Using Eq. (244), Eq. (243) may be rewritten to explicitly look like the Lagrangian of a massless spin-2 field:

$$\mathcal{L} = \underbrace{-\frac{c}{2} f^{\mu\nu} \partial^2 f_{\mu\nu}}_{\substack{\text{massless} \\ \text{spin-2 field}}} + \underbrace{\frac{c\sqrt{16\pi\hbar}}{p_p} f^{\mu\nu} \left(T_{\mu\nu} - \frac{1}{2} T_{\alpha\beta}\eta^{\alpha\beta}\eta_{\mu\nu} \right)}_{\substack{\text{interactions with} \\ \text{matter/energy } (T_{\mu\nu})}} + \underbrace{\mathcal{O}\left(\frac{\partial^2}{p_p^2} f^3\right) + \mathcal{O}\left(\frac{\partial^3}{p_p^3} f^4\right) + ...}_{\substack{\text{higher-order} \\ \text{(nonlinear) terms}}} \tag{247}$$

The higher-order terms in Eq. (247) may be interpreted in terms of either fields or particles:

- **Field interpretation.** The energy in a gravitational field is proportional to the square of the field's amplitude (just as this is true for electric or magnetic fields). To account for the coupling of the field to this energy of its own, a term of order f^2 could be added to the stress-energy tensor $T_{\mu\nu}$ in Eq. (247). This would result in a term of total order f^3 in the Lagrangian (since $T_{\mu\nu}$ gets multiplied by f), which is exactly what the first higher-order term is. However, the Lagrangian (kinetic energy minus potential energy) is closely related to the stress-energy tensor (components of energy and momentum), so the new $\mathcal{O}(f^3)$ term in the Lagrangian leads to a new $\mathcal{O}(f^3)$ term in $T_{\mu\nu}$. This latest term gets multiplied by f in Eq. (247), resulting in the next higher-order term $\mathcal{O}(f^4)$ in the Lagrangian. This cycle continues, generating higher and higher order terms in the Lagrangian.

- **Particle interpretation.** Gravitons couple to energy and gravitons carry energy, so they must couple to themselves. The $\mathcal{O}(f^3)$ term represents an interaction vertex with three graviton lines, for example a graviton entering and leaving the vertex and emitting or absorbing a new graviton at the vertex. As long as the total energy entering and leaving the vertex is conserved, there is no limit on how many gravitons can be emitted or absorbed in the same process. Thus there are also vertices with four graviton lines [$\mathcal{O}(f^4)$], five graviton lines [$\mathcal{O}(f^5)$], etc, as shown in Fig. 20(b). For the same reason, vertices coupling gravitons to other particles may include any number of graviton lines, as illustrated in Fig. 20(c).

Because of all the tensor indices running around, the vertex factors and propagators for gravitons contain many terms, each a nasty tensor expression involving different indices. This makes detailed calculations in quantum gravity almost impossible, even with computers. Nonetheless, it is easy to consider at least the qualitative form of the Feynman rules for quantum gravity:

1. The coupling constant or vertex factor in front of the three-graviton vertex in Eq. (247) is $\mathcal{O}(\partial^2/p_p^2) = \mathcal{O}(p^2/p_p^2)$, where p is typical of the momenta entering or leaving the vertex.

2. Likewise, vertices with four or more graviton lines introduce factors of $\mathcal{O}(p^3/p_p^3)$ or higher.

3. Similarly, vertices coupling gravitons to other types of particles have factors of at least $\mathcal{O}(p^2/p_p^2)$.

4. The propagator for internal (virtual) graviton lines is $D = \mathcal{O}(1/p^2)$, just like the propagators for other massless bosons such as spin-0 Klein-Gordon particles and spin-1 photons.

5. Integration over internal lines or loops introduces a factor of order $\int d^4p$.

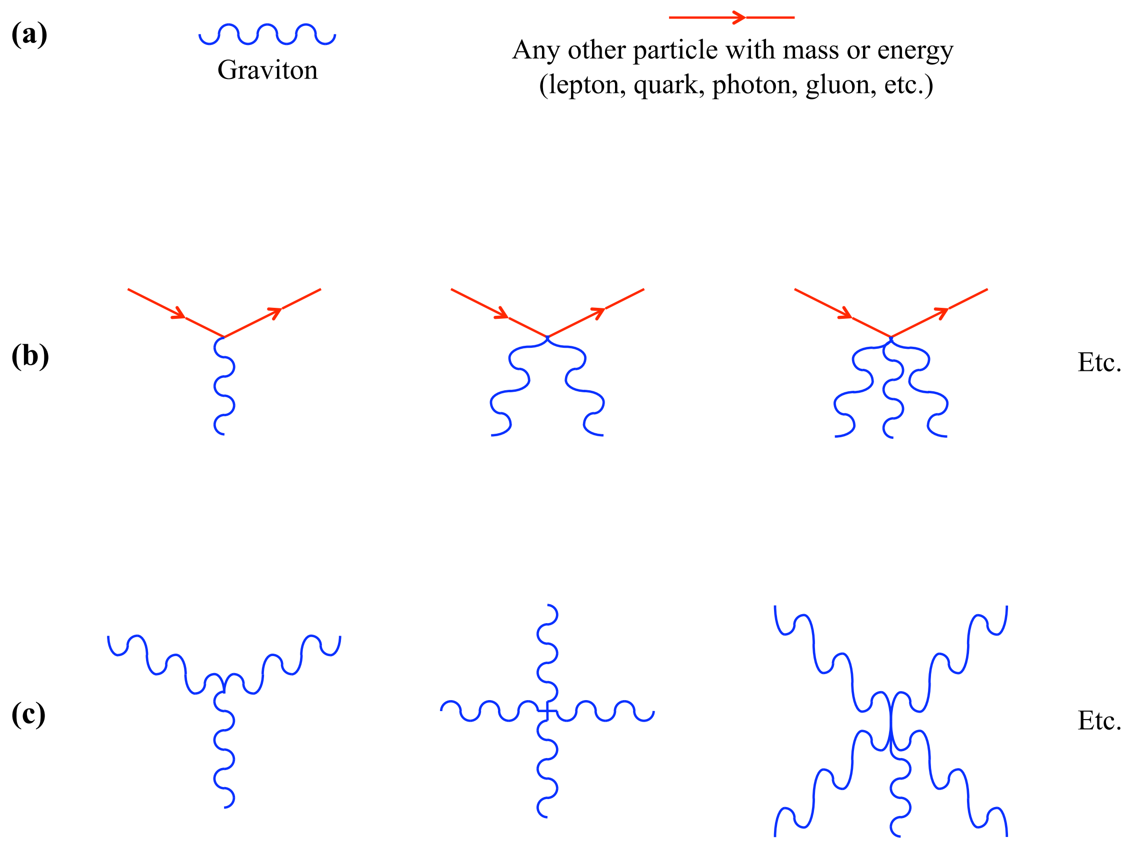

Figure 20. Fundamental components of Feynman diagrams in quantum gravity. (a) In this figure and Fig. 21, wavy lines represent gravitons and lines with arrows represent all other types of particles that have mass or energy (even photons, gluons, etc.). **(b)** Gravitons couple to particles that have mass or energy. There is no limit to how many gravitons may do this at a time, so there are vertices involving one, two, three, or any other number of gravitons. **(c)** Gravitons have energy, so they also couple to themselves. Again, there is no limit on the number of gravitons that may be involved, so there are vertices with three, four, five, or more graviton lines.

These qualitative Feynman rules may be used to determine if quantum gravity can be renormalized. For example, the graviton propagator must be modified to account for graviton loops, as shown in Fig. 21. In Fig. 10 and Eq. (141), a similar case was considered for loops in virtual photon lines in QED. By analogy with the QED case, a simple graviton propagator introduces a factor of D and a graviton loop yields a factor of

$$\Pi \;=\; \mathcal{O}\left[\left(\frac{p}{p_p}\right)^2\left(\int d^4p\frac{1}{p^2}\frac{1}{p^2}\right)\left(\frac{p}{p_p}\right)^2\right] = \mathcal{O}\left[\left(\frac{p}{p_p}\right)^4\right] \tag{248}$$

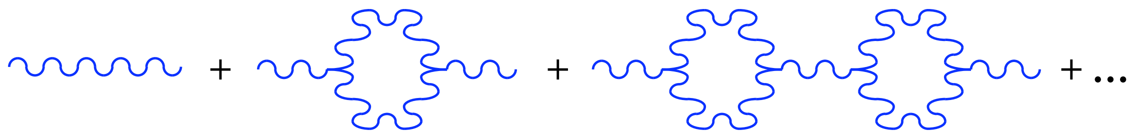

Figure 21. Graviton loop corrections to a virtual graviton line. The virtual graviton line is assumed to be an internal line in some larger Feynman diagram in quantum gravity. Note that loop diagrams involving other particles or vertices with more than three graviton lines are not shown here.

Thus the sum over the graphs in Fig. 21 is

$$\begin{aligned}\sum(\text{graviton graphs}) &= D + D\Pi D + D\Pi D\Pi D + ... \\ &= \mathcal{O}\left[\frac{1}{p^2}\right] + \mathcal{O}\left[\ln\left(\frac{p}{p_p}\right)\right] + \mathcal{O}\left[\left(\frac{p}{p_p}\right)^2\right] + \mathcal{O}\left[\left(\frac{p}{p_p}\right)^4\right] + ...\end{aligned} \tag{249}$$

In the limit of large momentum ($p/p_p \to \infty$), this result contains an infinite number of infinities, each worse than the last, and perturbation theory completely breaks down. And we haven't considered loop diagrams having vertices that involve other types of particles or more than three graviton lines. These families of loop diagrams introduce even more infinities.

In QED, loop diagrams only introduced two infinite quantities, which were swept under the rug by adjusting the definitions of mass and charge. As we have seen, loops in quantum gravity lead to an infinite number of infinities. Even if we could eliminate them all by adjusting the definitions of an infinite number of natural parameters, no useful predictions of physical effects could be made by a theory with an infinite number of fudge factors.

There are also other difficulties with quantum gravity. Splitting the metric into a background part that is treated classically and a perturbation component that is treated with quantum field theory seems very artificial and forced. It is the total metric that determines what parts of spacetime are in the past or the future and what regions cannot communicate with each other without exceeding the speed of light. Yet that is difficult to ascertain when the metric is split into two parts and one part is allowed to vary freely. Another way of viewing these difficulties is that in quantum gravity, the metric (or at least part of it) represents both the field and the total spacetime background in which that field occurs. This is in sharp contrast with the field theories for all the other fundamental forces, where the field in question and the spacetime background were completely separate quantities and the spacetime background was kept fixed. For more information on quantum gravity and its many headaches, see [7] and the references cited therein.

The bottom line is that it should be okay to treat gravity as a classical field via the theory of general relativity for momenta/energies less than the Planck momentum/energy and distances/times greater than the Planck length/time:

$$\begin{aligned} E_P &\equiv \sqrt{\hbar c^5/G} \approx 1.22 \cdot 10^{19}\ \text{GeV} \quad &&\textbf{Planck energy} \\ L_P &\equiv \sqrt{\hbar G/c^3} \approx 1.62 \cdot 10^{-33}\ \text{cm} \quad &&\textbf{Planck length} \\ T_P &\equiv \sqrt{\hbar G/c^5} \approx 5.39 \cdot 10^{-44}\ \text{sec} \quad &&\textbf{Planck time} \end{aligned} \tag{250}$$

Note that these values are many, many orders of magnitude beyond what can be achieved with existing technology. Indeed, only phenomena such as the initial moment of the big bang or the singularity of a black hole would go beyond these values. For such phenomena, general relativity and simple quantum gravity calculations break down and a new theory is needed to describe what happens. There are several candidate theories such as supergravity and superstrings [3], but one cannot tell which (if any) of them is correct unless they make unique predictions that can be tested in a realizable experimental system.

Even in cases where a gravitational field can be treated in a well-understood, classical manner, it can still have interesting interactions with electromagnetic or other fields that are behaving quantum mechanically. Quantum effects can be greatly altered if they occur in the curved spacetime of a strong gravitational field rather than the flat spacetime that is normally assumed [8]. In principle, one could calculate how the metric $g_{\mu\nu}$ of a curved spacetime would enter into the general Feynman rules for QED or other processes, although in practice this is so difficult that it isn't really done.

The best-know example of a quantum effect occuring in curved spacetime is Hawking radiation from black holes. For a simple description and calculation of this effect, see *Relativity* 6.1.

References

[1] D. Griffiths, *Introduction to Elementary Particles* (2nd edition, Wiley, New York, 2008). Like Kittel's *Introduction to Solid State Physics* and *Thermal Physics*, this book is an excellent model of what textbooks should be but rarely are–wonderfully readable in a clear and concise way, broad in scope yet full of numerous detailed practical examples, and founded on physical insight as well as equations. It reviews special relativity and nonrelativistic quantum mechanics, then goes on to cover spinless "toy" relativistic quantum theories, QED, weak interactions, and QCD. Its major weaknesses are that it does not show exactly where the Feynman rules come from, shies away from the scary math needed to take most QED calculations to their conclusion, and omits any discussion of quantum gravity. This summary of field theory has borrowed heavily from Griffiths' derivations and explanations while attempting to correct for these weaknesses.

[2] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, *Quantum Electrodynamics* (2nd ed., Pergamon Press, 1982). This book shares the characteristic strengths and weaknesses of the other texts in the Landau and Lifshitz series. It is terse yet exhaustive in its treatment of its chosen subject, and it contains many examples and insights that simply cannot be found elsewhere. On the other hand, it champions mathematics over physical explanations and uses a somewhat different notation than most Western textbooks. Considering the detail that the authors lavish on spinless quantum field theories and QED, it is very regrettable that they did not also address the weak interactions, QCD, and quantum gravity.

[3] M. Kaku, *Quantum Field Theory: A Modern Introduction* (Oxford U. Press, 1993). This is essentially the only field theory book that covers all four fundamental forces, and it is actually fairly decent. Its coverage of the material is somewhat uneven; for example, it gives an amazingly clear and thorough presentation of QED renormalization, yet it presents only a brief overview of most of QCD. It includes useful introductions to more advanced (and speculative) field theories, including grand unified theories, supersymmetry, and superstrings. Be warned that superstring theory is Kaku's real passion, so you should take his enthusiasm for it with several grains of salt.

[4] W. Greiner and J. Reinhardt, *Quantum Electrodynamics* (2nd ed., Springer, 1994).

[5] W. Greiner and B. Müller, *Gauge Theory of Weak Interactions* (2nd ed., Springer, 1996).

[6] W. Greiner and A. Schäfer, *Quantum Chromodynamics* (Springer, 1994).
Attempting to outdo the 10-volume Russian physics series by Landau and Lifshitz, the Germans have written a zillion-volume physics series, with at least 10 volumes just on quantum physics. That seems a bit excessive, but these three volumes provide more thorough coverage of the fundamental forces than can be found in other field theory books that try to cover all the forces in one book. They calculate just about everything that can be calculated, and they show every single step–pages and pages and pages of them. Therefore, while these books would make tedious introductions to their subjects, they are useful references for the gory details of many calculations.

[7] R. P. Feynman, *Feynman Lectures on Gravitation* (Addison-Wesley, 1995). While not as clear and insightful as *The Feynman Lectures on Physics*, this book may serve as an introductory textbook on quantum gravity, especially since there aren't really any other books competing for that job. At least as important as Feyman's actual presentation, the book also contains extensive introductions and a bibliography covering major developments in quantum gravity since Feynman worked on the problem in the early 1960s.

[8] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge U. Press, 1982). This is yet another book which is recommended chiefly because of its lack of competitors. It does indeed describe how quantum field theory changes when it occurs in curved instead of flat spacetime. However, it devotes surprisingly little attention to unique physical effects which should result, and it does not spell out how curved spacetime affects the Feynman rules so that readers may work out physical effects themselves. Perhaps that is because little is actually known about those areas. Useful advances in understanding may have occurred since this book was written.