Heat Transfer

Todd H. Rider **thor@riderinstitute.org** **13 November 2019**

Any suggestions for improvements would be greatly appreciated.

I saw many herds of naked souls, who were all lamenting very miserably; and there seemed imposed upon them a diverse law.

Some were lying supine upon the ground; some sitting all crouched up; and others roaming incessantly.

Those that moved about were much more numerous; and those that were lying in the torment were fewer, but uttered louder cries of pain.

Over all the great sand, falling slowly, rained dilated flakes of fire, like those of snow in Alps without a wind.

As the flames which Alexander, in those hot regions of India, saw fall upon his host, entire to the ground;

whereat he with his legions took care to tramp the soil, for the fire was more easily extinguished while alone:

so fell the eternal heat, by which the sand was kindled, like tinder under flint and steel, redoubling the pain.

Ever restless was the dance of miserable hands, now here, now there, shaking off the fresh burning.

–Dante Alighieri, *The Inferno*, Canto XIV (ca. 1320)

...it gives me Occasion to mention some loose Notions relating to Heat and Cold, which I have for some Time entertain'd, but not yet reduc'd into any Form. Allowing common Fire as well as the Electrical, to be a Fluid, capable of permeating other Bodies, and seeking an Equilibrium, I imagine some Bodies are better fitted by Nature to be Conductors of that Fluid than others; and that generally those which are the best Conductors of the Electrical Fluid, are also the best Conductors of this...

–Benjamin Franklin, letter to John Lining (April 14, 1757)

Overview

Thermal energy may be transferred from one place to another by several different mechanisms, including conduction, forced convection, natural convection, condensation, boiling, and thermal radiation. While heat transfer processes are sometimes so complicated that they must be measured experimentally and expressed as empirical correlations, the emphasis here will primarily be on developing simple physical models and analytical expressions for the various heat transfer mechanisms. Applications of heat transfer technology include: thermal energy transport in heat engines and refrigerators; cooling of integrated circuit chips, nuclear reactors, and engines; and insulation of everything from thermos bottles to buildings.

Contents

1 Conduction

As illustrated in Fig. 1, thermal energy may be transferred from one place to another by several different mechanisms, including (a) conduction, (b) forced convection, (c) natural convection, (d) condensation, (e) boiling, and (f) thermal radiation. This section will focus on conduction, in which heat is transferred through matter by diffusion. The remaining forms of heat transfer will be covered in order in Sections 2-6.

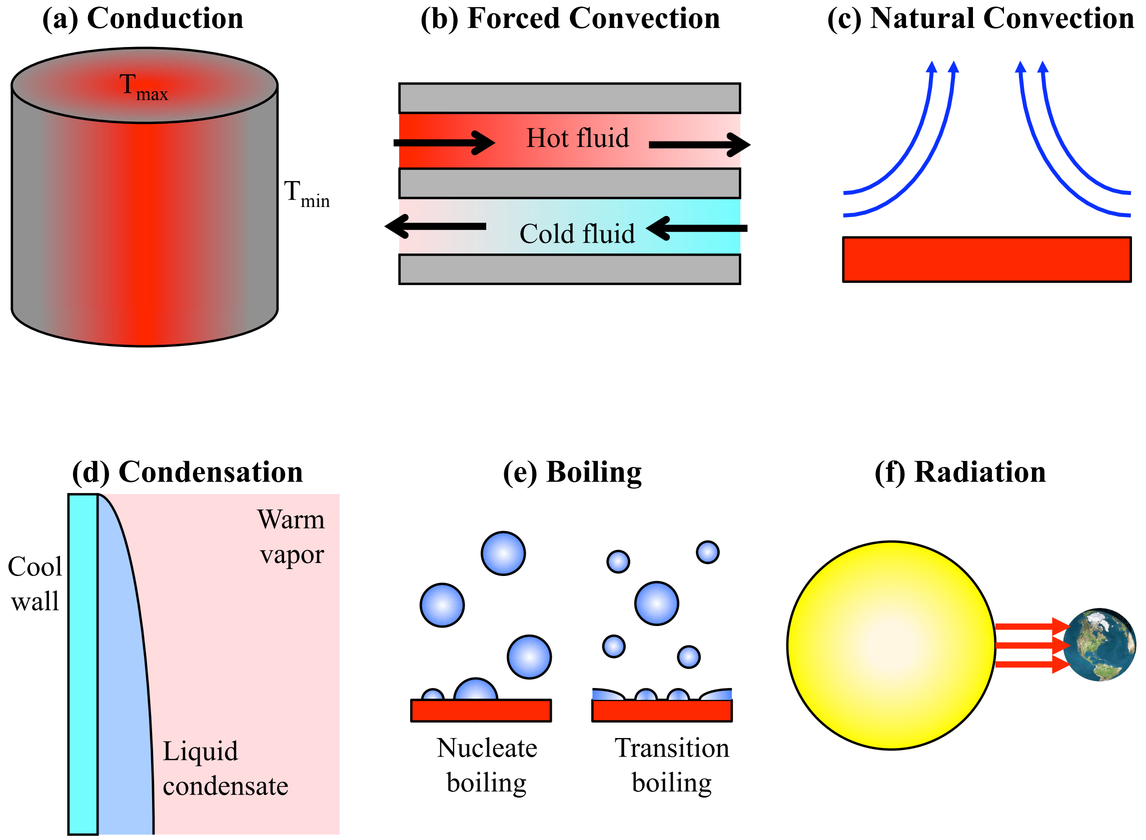


Figure 1. Modes of heat transfer include (a) conduction, (b) forced convection, (c) natural convection, (d) condensation, (e) boiling, and (f) thermal radiation.

1.1 Fundamentals of Conduction

In this and the following sections, Q is defined to be the rate of heat energy flow; it is measured in Watts. The heat flux due to a heat flow Q spread over a cross-sectional area of A is $q \equiv Q/A$, which has units of Watts/m^2.

Since heat energy flows from hot regions to cold regions, and since a greater temperature difference leads to a greater heat flow, the heat flux in the presence of a temperature gradient ∇T is

$$\mathbf{q} \;=\; -\kappa \nabla T\;, \qquad \textbf{Fourier's law of conduction} \tag{1}$$

in which κ is the thermal conductivity [W/m^{o}K] of the medium through which the heat is flowing.

Conductive heat transfer is due to three separate mechanisms:

1. Thermal energy may be carried by electrons, if the electrons are free to move throughout a substance, as is the case with metals and plasmas.
2. Heat energy may also be carried by vibrations in the substance (called phonons); this is especially significant if the substance is arranged into an organized solid lattice which can vibrate.
3. If the substance is an amorphous solid or a fluid, heat energy may also be carried by particles of the fluid which randomly migrate through the fluid and transfer their energy to other particles within the fluid by collisions. This is a less efficient mechanism of heat transfer and thus accounts for the generally lower thermal conductivities of substances that have neither free electrons nor lattices.

Because solid metals have both free electrons and a lattice structure, they generally have large thermal conductivities. Table 1 lists the thermal conductivities and their explanations for a wide range of materials. For a more detailed look at the mechanisms of conductive heat transfer, see *Solid State Physics*, *Materials Science*, and *Plasma Physics and Fusion*.

Material	**κ (W/m^{o}K)**	**Explanation**
Diamond	1300	Nonmetal but great lattice
Copper	400	Metal
Aluminum	240	Metal
Iron	80	Metal
Steel	20	Metal but not pure
Mercury	8	Metal but no lattice
Ceramic	4	Lattice
Glass	1	No lattice
Wood	0.1	No lattice
Cork	0.04	Air pockets

Table 1. Thermal conductivity κ for various materials.

The higher a material's absolute temperature T (in degrees Kelvin), the more thermal energy (in Joules) that material has. From *Statistical Physics* ?.? and *Thermodynamics and Propulsion* ?.?, the specific heat capacity at constant pressure for a material is c_p [J/kg^{o}K]. If the mass density of the material is ρ [kg/m^3], the heat capacity will be $C_p \equiv \rho c_p$ [J/^{o}K], and the density u [J/m^3] of thermal energy in the material will be

$$u \;=\; C_p T \;=\; \rho c_p T \tag{2}$$

The total change in the thermal energy density, the change due to heat diffusion, and the change due to heat sources or heat sinks may be written as:

$$\left(\frac{du}{dt}\right)_{\text{total}} = C_p\frac{\partial T}{\partial t} = \rho c_p\frac{\partial T}{\partial t} \quad \textbf{net local increase of thermal energy} \tag{3}$$

$$\left(\frac{\partial u}{\partial t}\right)_{\text{diffusion}} = -\nabla\cdot\mathbf{q} = \kappa\nabla^2 T \quad \textbf{heat diffusing into/out of control volume} \tag{4}$$

$$\left(\frac{\partial u}{\partial t}\right)_{\text{other}} \equiv S \quad \textbf{other (explicitly defined) thermal sources/sinks} \tag{5}$$

Putting these three terms together, the conservation equation for thermal energy is:

$$\begin{aligned}\left(\frac{du}{dt}\right)_{\text{total}} &= \left(\frac{\partial u}{\partial t}\right)_{\text{diffusion}} + \left(\frac{\partial u}{\partial t}\right)_{\text{other}}\\ \rho c_p\frac{\partial T}{\partial t} &= \kappa\nabla^2 T + S\\ \frac{\partial T}{\partial t} &= \alpha\nabla^2 T + \frac{S}{\rho c_p}\,, \end{aligned} \tag{6}$$

where the **thermal diffusivity** $\alpha \equiv \kappa/\rho c_p$ [m^2/sec] is simply the diffusion constant for heat energy which is moving around via thermal conduction.

Some special cases of Eq. (6) deserve mention:

- If the system is in steady state ($\partial T/\partial t = 0$) without explicit thermal sources or sinks ($S = 0$), Eq. (6) assumes the form of Laplace's equation. This case will be considered in Section 1.2.
- If the system is in steady state ($\partial T/\partial t = 0$) but thermal sources or sinks are present ($S \neq 0$), Eq. (6) reduces to the form of Poisson's equation, as will be discussed in Section 1.3.
- If the system is not in steady state ($\partial T/\partial t \neq 0$) and there are no sources or sinks ($S = 0$), Eq. (6) reduces to a simple diffusion equation, as will be shown in Section 1.4.

Whatever form Eq. (6) takes, in constructing $T(\mathbf{x})$ solutions for it for adjoining regions, at the boundary between the regions one must match both T (due to continuity of temperature) and $\hat{\mathbf{n}}\cdot\nabla T$ (due to continuity of heat flux, with $\hat{\mathbf{n}}$ defined as the unit vector normal to the boundary). Three different boundary conditions on T which are often used with Eq. (6) are:

1. T may be a known constant at a boundary.
2. A surface may be adiabatic, so that $\mathbf{q}\cdot\hat{\mathbf{n}} = 0$, or $\hat{\mathbf{n}}\cdot\nabla T = 0$, at the surface.
3. The heat flux at a boundary may be a fixed nonzero constant, so $\hat{\mathbf{n}}\cdot\nabla T$ at the boundary will be a nonzero constant.

1.2 Steady State with No Internal Heat Generation

For the steady state with no sources or sinks, Eq. (6) takes the form of Laplace's equation:

$$\nabla^2 T = 0 \qquad \text{Laplace's equation (assumes } S = 0 \text{ and } \partial T/\partial t = 0\text{)} \tag{7}$$

The heat flow $Q_{i \to o}$ from the inside to the outside of a hollow shell may be analyzed for various geometries. It will be assumed that T_i is the temperature at the inner surface of the shell and T_o is the temperature at the outer surface. By defining a thermal resistance R for the shell, the heat flow may be written in a form similar to Ohm's law ($I = \Delta V/R$ for an electric current I analogous to $Q_{i \to o}$, voltage difference ΔV comparable to $T_i - T_o$, and electrical resistance R analogous to the thermal resistance), as illustrated in Fig. 2(a):

$$Q_{i \to o} \;=\; \frac{T_o - T_i}{R} \tag{8}$$

For a slab of thickness L and area A, as shown in Fig. 2(b), Eqs. (7) and (1) may be solved to find the temperature profile, heat flow, and thermal resistance:

$$\frac{d^2T}{dx^2} \;=\; 0 \quad \to \quad T(x) \;=\; T_i + (T_o - T_i)\frac{x}{L} \tag{9}$$

$$Q_{i \to o} \;=\; -\kappa A \left[\frac{dT(x)}{dx}\right]_{x=0} \;=\; \frac{\kappa A}{L}(T_o - T_i) \;=\; \frac{T_o - T_i}{R_{\text{slab}}} \tag{10}$$

$$R_{\text{slab}} \;\equiv\; \frac{L}{\kappa A} \tag{11}$$

Similarly, for a cylindrical shell with inner radius r_i, outer radius r_o, and length $L \gg r_o$, as illustrated in Fig. 2(c), one finds:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) \;=\; 0 \quad \to \quad T(r) \;=\; T_i + (T_o - T_i)\frac{\ln(r/r_i)}{\ln(r_o/r_i)} \tag{12}$$

$$Q_{i \to o} \;=\; -\kappa(2\pi r_i L)\left[\frac{dT(r)}{dr}\right]_{r=r_i} \;=\; \frac{2\pi L\kappa}{\ln(r_o/r_i)}(T_o - T_i) \;=\; \frac{T_o - T_i}{R_{\text{cyl}}} \tag{13}$$

$$R_{\text{cyl}} \;\equiv\; \frac{\ln(r_o/r_i)}{2\pi L\kappa} \tag{14}$$

For a spherical shell with inner radius r_i and outer radius r_o, as shown in Fig. 2(d), the results are:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) \;=\; 0 \quad \to \quad T(r) \;=\; T_i + (T_o - T_i)\left(\frac{1/r_i - 1/r}{1/r_i - 1/r_o}\right) \tag{15}$$

$$Q_{i \to o} \;=\; -\kappa(4\pi r_i^2)\left[\frac{dT(r)}{dr}\right]_{r=r_i} \;=\; 4\pi\kappa\left(\frac{1}{r_i} - \frac{1}{r_o}\right)^{-1}(T_o - T_i) \;=\; \frac{T_o - T_i}{R_{\text{sphere}}} \tag{16}$$

$$R_{\text{sphere}} \;\equiv\; \frac{1}{4\pi\kappa}\left(\frac{1}{r_i} - \frac{1}{r_o}\right) \tag{17}$$

For more information on solutions of Laplace's equation, see *Applied Mathematics* ?.?.

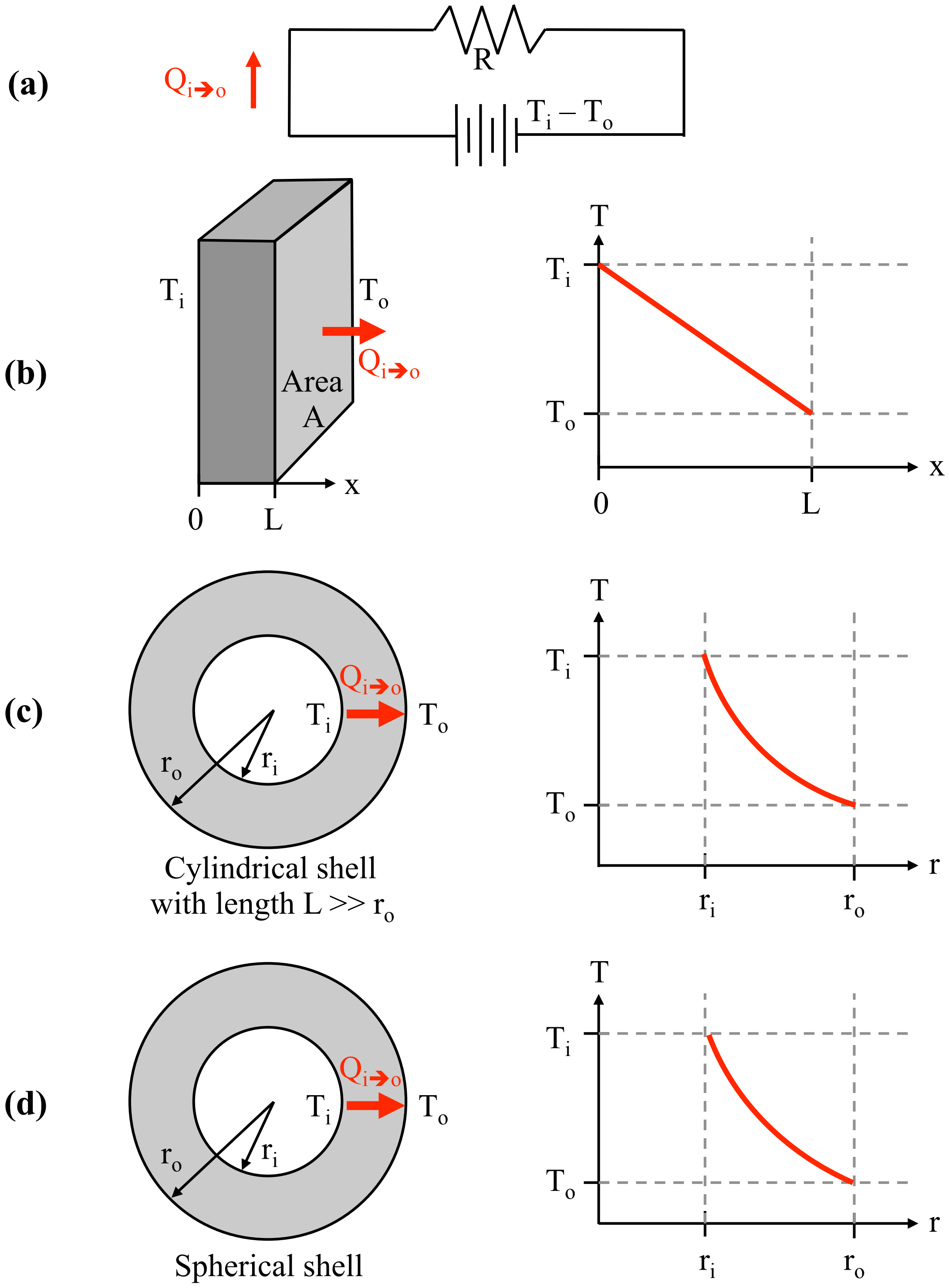


Figure 2. Conductive heat transfer in simple geometries with no internal heat generation. **(a)** The heat flow $Q_{i \to o}$ may be modelled as an electric current driven by a "voltage" difference $T_i - T - o$ across a resistance R. Temperature gradients across **(b)** a slab, **(c)** a cylindrical shell, and **(d)** a spherical shell drive heat flow.

In certain cases the boundary between two substances also has a thermal resistance. This may be a contact resistance between two adjacent solid objects or a convective resistance between a solid surface and a fluid. These situations may be described by a heat transfer coefficient h:

$$q = h\Delta T \qquad \textbf{Newton's law of cooling} \tag{18}$$

in which ΔT is the temperature difference across the boundary.

For cases described by Eq. (18), the thermal resistance of the boundary may be defined as

$$R_{\text{boundary}} = \frac{1}{hA}\,, \tag{19}$$

where A is the area of the boundary.

Thermal resistances which are placed in series [Fig. 3(a)] or parallel [Fig. 3(b)] arrangements may be analyzed in exactly the same fashion as series and parallel electrical resistance networks. Thermal resistances R_1 and R_2 in series are equivalent to one resistance R_1+R_2, whereas resistances R_1 and R_2 in parallel are equivalent to one resistance $R_1R_2/(R_1+R_2)$.

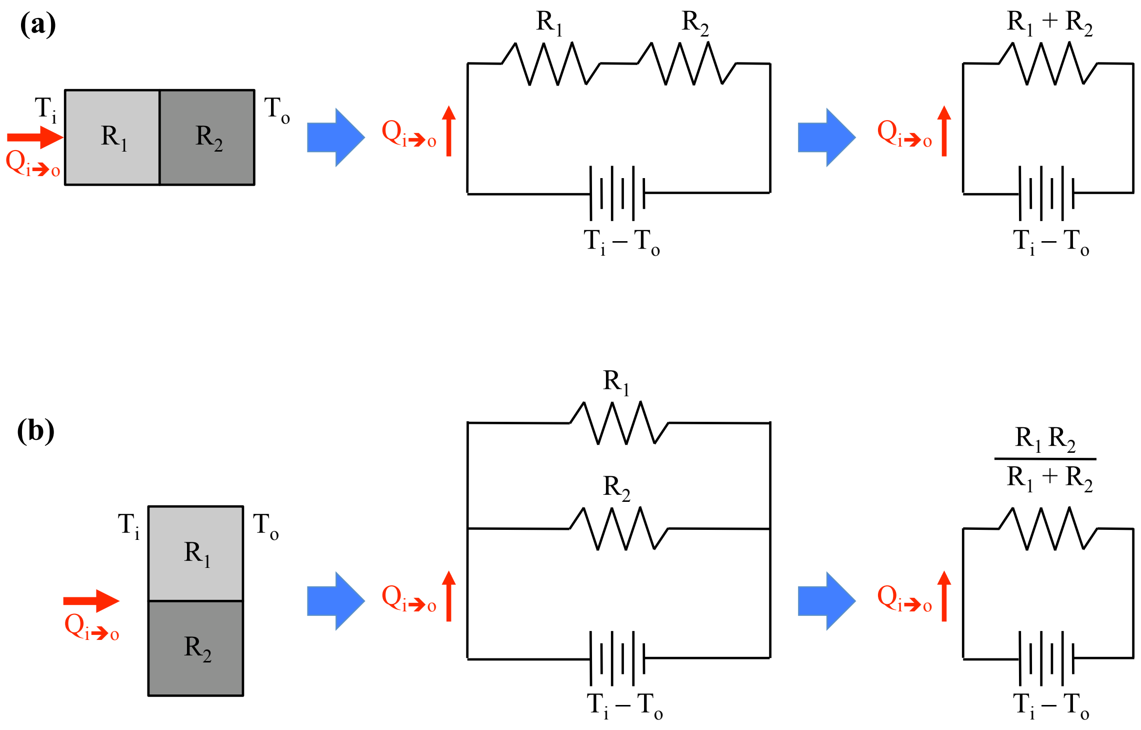


Figure 3. Thermal resistances in series and parallel behave like electrical resistances in circuits. (a) Thermal resistances R_1 and R_2 in series are equivalent to one resistance $R_1 + R_2$. **(b)** Resistances R_1 and R_2 in parallel are equivalent to one resistance $R_1R_2/(R_1 + R_2)$.

1.3 Steady State with Uniform Internal Heat Generation

For steady state systems with thermal sources/sinks, Eq. (6) takes the form of Poisson's equation:

$$\nabla^2 T \quad = \quad -\frac{S}{\kappa} \qquad\qquad \text{Poisson's equation (assumes } \partial T/\partial t = 0) \tag{20}$$

Some cases of Poisson's equation for heat conduction will be solved as illustrations, as shown in Fig. 4. For simplicity it will be assumed that S represents uniform internal heat generation and that the temperature at the surface of the substance in question is T_{outside}.

For a slab extending from $x = -a$ to $x = +a$ and infinite in the y and z directions, as shown in Fig. 4(a), Eq. (20) yields:

$$\frac{d^2T}{dx^2} \quad = \quad -\frac{S}{\kappa} \qquad \rightarrow \qquad T(x) - T_{\text{outside}} \quad = \quad \frac{1}{2}\,\frac{S}{\kappa}\left(a^2 - x^2\right) \tag{21}$$

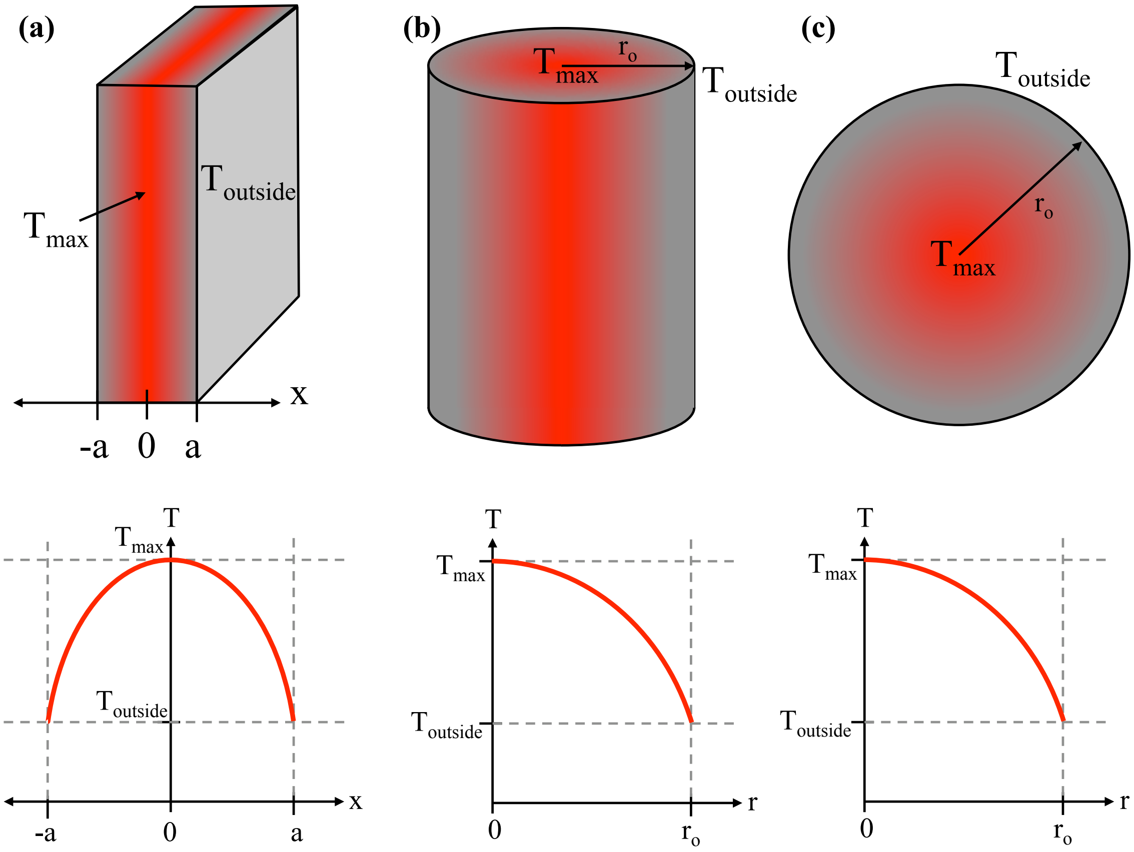


Figure 4. Temperature profiles in a (a) slab, (b) cylinder, and (c) sphere with uniform internal heat generation.

Likewise, for an infinitely long cylinder with radius r_o, as depicted in Fig. 4(b), the temperature profile is:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{S}{\kappa} \quad \rightarrow \quad T(r) - T_{\text{outside}} = \frac{1}{4}\frac{S}{\kappa}\left(r_o^2 - r^2\right) \tag{22}$$

For a sphere of radius r_o, as illustrated in Fig. 4(c), Eq. (20) produces the result:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{S}{\kappa} \quad \rightarrow \quad T(r) - T_{\text{outside}} = \frac{1}{6}\frac{S}{\kappa}\left(r_o^2 - r^2\right) \tag{23}$$

For fancier solutions of Poisson's equation, see *Applied Mathematics* ?.?.

As an example of how all of this stuff may be used, consider the configuration in Fig. 5. A cylindrical nuclear fuel rod is surrounded by inert cladding material, which is in turn surrounded by cooling fluid.

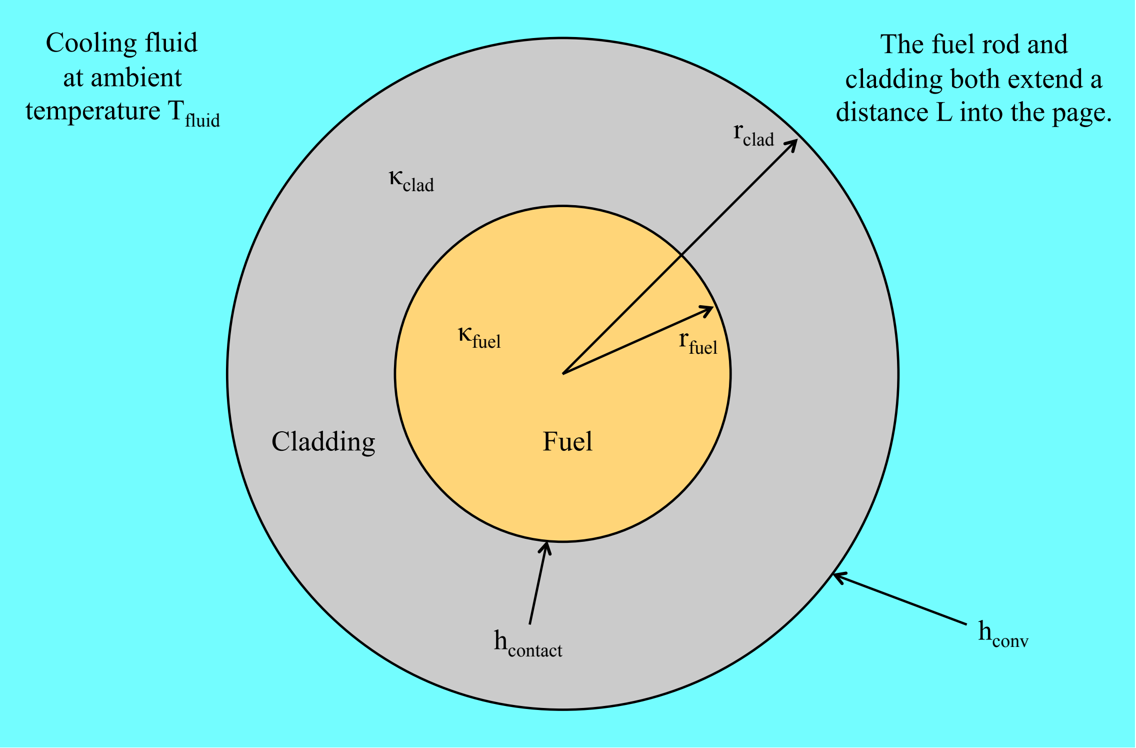


Figure 5. A nuclear fuel rod surrounded by an inert cladding and cooling fluid. There is a thermal contact resistance between the fuel and the cladding, and a convective resistance between the surface of the cladding and the cooling fluid.

Consider the various sources of temperature drops between the center of the fuel rod and the ambient cooling fluid:

1. If the fuel rod has uniform heat generation S, thermal conductivity κ_{fuel}, radius r_{fuel}, and length L, the temperature drop from the center to the outside of the fuel rod is given by Eq. (22) as $T(0) - T(r_{\text{fuel}}) = S r_{\text{fuel}}^2 / 4\kappa_{\text{fuel}}$.
2. From Eq. (19), there is a contact resistance $R_{\text{contact}} = 1/(2\pi r_{\text{fuel}} L h_{\text{contact}})$ between the fuel and insulation.
3. If the fuel rod is surrounded by an insulating cladding layer of conductivity κ_{clad}, inner radius r_{fuel}, and outer radius r_{clad}, the thermal resistance of this insulation as given by Eq. (14) is $R_{\text{clad}} = \ln(r_{\text{clad}}/r_{\text{fuel}})/(2\pi L \kappa_{\text{clad}})$.
4. From Eq. (19), there is a convective resistance $R_{\text{conv}} = 1/(2\pi r_{\text{clad}} L h_{\text{conv}})$ between the insulation and the bulk of the surrounding fluid.

In the steady state the heat flow out of the insulated rod must be equal to the total heat generated in the fuel rod, $Q = \pi r_{\text{fuel}}^2 L S$. Thus the temperature at the center of the fuel rod relative to the fluid temperature T_{fluid} is

$$\begin{aligned} T(0) - T_{\text{fluid}} &= [T(0) - T(r_{\text{fuel}})] \; + \; [T(r_{\text{fuel}}) - T_{\text{fluid}}] \\ &= \frac{1}{4}\,\frac{S}{\kappa_{\text{fuel}}}\, r_{\text{fuel}}^2 \; + \; Q\,(R_{\text{contact}} + R_{\text{clad}} + R_{\text{conv}}) \\ &= \frac{1}{4}\,\frac{S}{\kappa_{\text{fuel}}}\, r_{\text{fuel}}^2 \; + \; (\pi r_{\text{fuel}}^2 L S)\left[\frac{1}{2\pi r_{\text{fuel}} L h_{\text{contact}}} + \frac{\ln(r_{\text{clad}}/r_{\text{fuel}})}{2\pi L \kappa_{\text{clad}}} + \frac{1}{2\pi r_{\text{clad}} L h_{\text{conv}}}\right] . \end{aligned} \tag{24}$$

1.4 Non-Steady State with No Internal Heat Generation

If there are no explicit thermal sources or sinks, Eq. (6) reduces to a simple diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad \text{standard diffusion equation } (S = 0) \tag{25}$$

[For the diffusion of heat in a fluid which has flow velocity **v**, Eq. (25) must be modified to include a convective derivative:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) T = \alpha \nabla^2 T \; . \tag{26}$$

The convective derivative accounts for variations which are purely due to time and also variations due to fluid motion downstream where conditions are different (see *Fluid Mechanics and Aerodynamics* ?.? for more information).]

For an object which is not in the steady state, it is important to determine if the time scale of the object's temperature variations is dominated by either conduction within the object or heat transfer at the surface of the object, or if both phenomena are important. For illustrative purposes, an object subjected to convective cooling at its outer surface will be considered here, but this analysis may be readily generalized to include other heat transfer mechanisms at the object's surface. From Newton's law of cooling, the heat transfer coefficient h determines the convective heat flux between the object and its surroundings, $q_{\text{conv}} \sim h\Delta T$, where ΔT is a characteristic temperature difference for the system. On the other hand, Fourier's law of conduction indicates that the conductive heat flux through the object will vary like $q_{\text{cond}} \sim \kappa \Delta T / L_c$, in which a characteristic length $L_c \equiv V/A$ has been defined for the object in terms of the object's volume V and surface area A. In order to weigh the relative importance of these two effects, it is conventional to take their ratio and define it as the **Biot number**:

$$\text{Bi} \equiv \frac{hL_c}{\kappa} \sim \frac{q_{\text{conv}}}{q_{\text{cond}}} \; . \tag{27}$$

For $\text{Bi} \ll 1$, conductive heat flow through the object proceeds much more rapidly than convective heat transfer at the object's surface. In this case one may use the **lumped capacitance model**, which assumes that the temperature in the object is uniform and that the time constant τ for temperature changes is determined by surface convection. For this case, one finds:

$$\rho c_p V \frac{d(T - T_\infty)}{dt} = -hA(T - T_\infty) \; , \tag{28}$$

where the object has mass density ρ, specific heat capacity c_p, volume V, temperature T, and area A. The temperature of the surrounding fluid far from the object is T_∞.

If the initial temperature of the object at time $t = 0$ is T_i, the solution of Eq. (28) is

$$T(t) - T_\infty = (T_i - T_\infty) e^{-t/\tau} \tag{29}$$

$$\text{with } \tau \equiv \frac{\rho c V}{hA} \tag{30}$$

In their eternal quest to make things seem more mysterious by redefining everything in terms of dimensionless numbers, heat transferists have defined a dimensionless **Fourier number** Fo, which may be regarded as the ratio of the object's heat conduction rate to its thermal energy storage rate:

$$\mathrm{Fo} \;\equiv\; \frac{\alpha t}{L_c^2} \;=\; \frac{\kappa t}{\rho c (V/A)^2} \tag{31}$$

Using the Biot and Fourier numbers, Eq. (29) may be rewritten in the more cryptic form,

$$T - T_\infty \;=\; (T_i - T_\infty) e^{-\mathrm{Bi}\,\mathrm{Fo}} \tag{32}$$

Enlightening, right? Thought so.

For $\mathrm{Bi} \gg 1$ and $\mathrm{Fo} < 0.05$, the slow pace of internal heat conduction is what determines the temperature variations of the object. In this case one may use the **semi-infinite solid model**, in which a "wavefront" of temperature change slowly creeps through the object. For a surface in the $x = 0$ plane with a constant surface temperature $T_s \equiv T(x = 0)$, the temperature profile which satisfies Eq. (25) in a semi-infinite solid occupying the region $x \geq 0$ is

$$T(x,t) - T_s \;=\; (T_i - T_s)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{33}$$

Using Eq. (33), the heat flux q_s at the object's surface is found to be

$$q_s \;\equiv\; -\kappa \left[\frac{\partial T}{\partial x}\right]_{x=0} \;=\; \frac{\kappa\,(T_s - T_i)}{\sqrt{\pi \alpha t}} \tag{34}$$

As shown in [1-3], the semi-infinite solid model may also be solved for boundary conditions other than constant surface temperature T_s, such as constant surface heat flux q_s.

For $\mathrm{Bi} \gg 1$ and $\mathrm{Fo} > 0.05$, or for $\mathrm{Bi} \sim 1$, the solutions are not as clean and simple. Results for these cases are given in [1-3] in the form of graphs and approximate series solutions.

1.5 Fins

To cool an object better, **fins** are often added. By increasing the surface area of the object, the fins can increase the amount of convective heat transfer from the object.

Consider a fin which extends out from an object in the x direction and has a length L, as shown in Fig. 6. For simplicity, assume that the cross-sectional area A and perimeter P in the $y - z$ plane, as well as the thermal conductivity κ of the fin material, are constant all along the length of the fin.

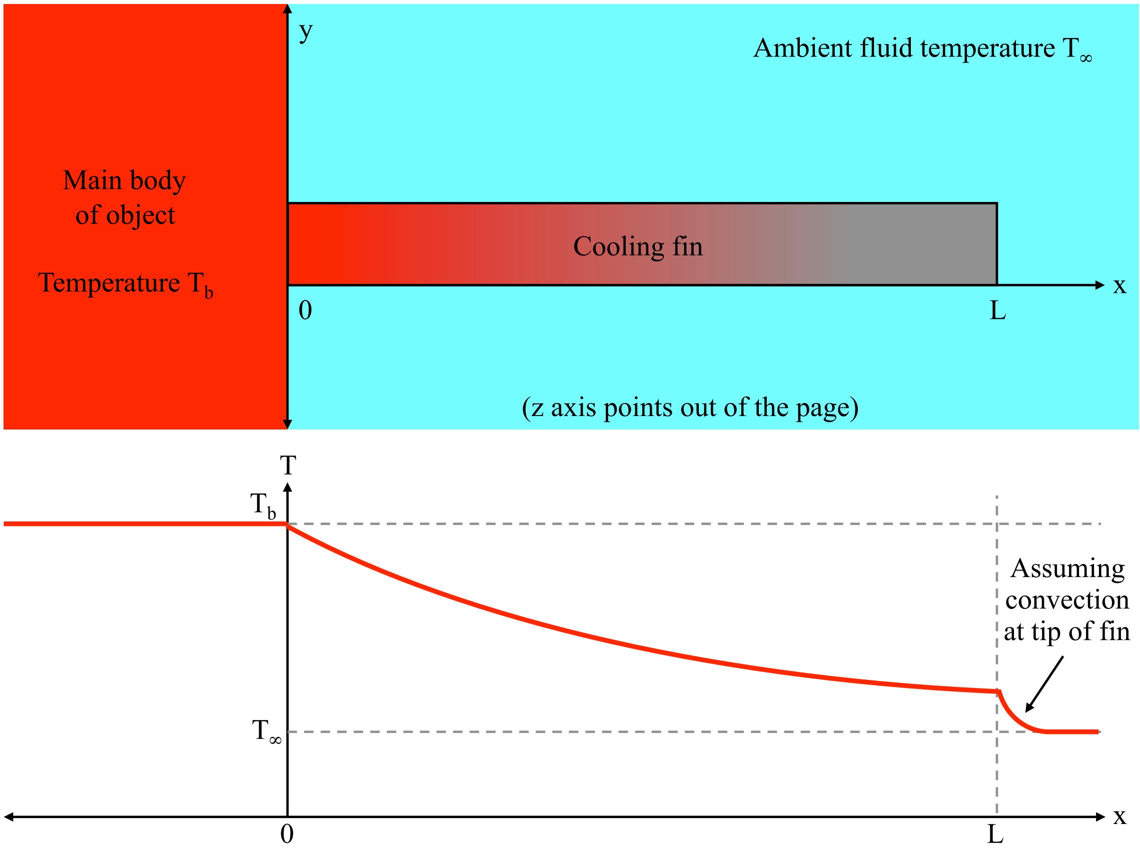


Figure 6. A body with base temperature T_b has a cooling fin that protrudes into a fluid with an ambient temperature of T_∞. The temperature steadily declines along the length of the fin. The precise solution at the end depends on the boundary conditions at the fin's tip.

If the fin is surrounded by a medium which has a temperature T_∞ far from the fin and h is the coefficient of heat transfer between the fin and the surrounding medium, Eq. (20) becomes

$$\frac{d^2(T - T_\infty)}{dx^2} = \frac{hP}{\kappa A}\,(T - T_\infty)\ . \qquad (35)$$

Solutions of Eq. (35) are of the form

$$T(x) \; - \; T_\infty \;\; = \;\; C_1 e^{-mx} \; + \; C_2 e^{+mx} \quad , \tag{36}$$

$$\text{with} \quad m \;\; \equiv \;\; \sqrt{\frac{hP}{\kappa A}} \tag{37}$$

One needs to know the boundary conditions in order to determine the constants C_1 and C_2. Note that the temperature in the fin tends to vary on a length scale of $1/m$.

The total heat flow dissipated by the fin may be calculated by considering the rate of heat conduction across the base of the fin, the boundary between the main body of the object and the fin:

$$Q_{\text{fin}} \;\; = \;\; -\kappa A \left[\frac{dT(x)}{dx} \right]_{x=0} \quad . \tag{38}$$

Generally the temperature at the base of the fin is known, $T(x = 0) \equiv T_b$, and it is then necessary to find a boundary condition for the other end of the fin. **The simplest boundary condition is that the fin is infinitely long (or at least $mL \gg 1$)**, in which case

$$T(x) - T_\infty \;\; = \;\; (T_b - T_\infty) e^{-mx} \tag{39}$$

$$Q_{\text{fin}} \;\; = \;\; \kappa A m \; (T_b - T_\infty) \tag{40}$$

A different common boundary condition on the $x = L$ end of the fin is that it is insulated, which by Fourier's law implies that $[dT/dx]_{x=L} = 0$. This boundary condition leads to the expressions

$$T(x) - T_\infty \;\; = \;\; (T_b - T_\infty) \, \frac{\cosh[m(L-x)]}{\cosh(mL)} \tag{41}$$

$$Q_{\text{fin}} \;\; = \;\; \kappa A m \; \tanh(mL) \; (T_b - T_\infty) \tag{42}$$

If instead convection at the fin tip is taken into account with heat transfer coefficient h, the resulting expressions are:

$$T(x) - T_\infty \;\; = \;\; (T_b - T_\infty) \, \frac{\kappa m \cosh[m(L-x)] + h \sinh[m(L-x)]}{\kappa m \cosh(mL) + h \sinh(mL)} \tag{43}$$

$$Q_{\text{fin}} \;\; = \;\; \kappa A m \, \frac{\kappa m \sinh(mL) + h \cosh(mL)}{\kappa m \cosh(mL) + h \sinh(mL)} \; (T_b - T_\infty) \tag{44}$$

One may define the thermal resistance of a fin as

$$R_{\text{fin}} \;\; \equiv \;\; \frac{T_b - T_\infty}{Q_{\text{fin}}} \quad . \tag{45}$$

The **fin efficiency** η_{fin} is defined as the ratio of Q_{fin} to the fin's maximum possible heat dissipation (Q_{max}) which would occur if the entire surface area A_{fin} of the fin were at temperature T_b:

$$\eta_{\text{fin}} \equiv \frac{Q_{\text{fin}}}{Q_{\text{max}}} = \frac{Q_{\text{fin}}}{hA_{\text{fin}}(T_b - T_\infty)} \tag{46}$$

For a fin with an insulated tip, $A_{\text{fin}} = LP$ and $\eta_{\text{fin}} = \tanh(mL)/mL$. For a fin with an uninsulated tip, $A_{\text{fin}} = LP + A$.

Similarly, the **surface efficiency** η_0 of an object with a fin is defined as the ratio of Q_{fin} to the object's maximum possible total heat dissipation ($Q_{\text{total, max}}$) which would occur if the entire surface area A_{total} of the object were at the temperature T_b:

$$\begin{aligned}\eta_0 &\equiv \frac{Q_{\text{fin}}}{Q_{\text{total, max}}} = \frac{Q_{\text{fin}}}{hA_{\text{total}}(T_b - T_\infty)} \\ &= 1 - \frac{A_{\text{fin}}}{A_{\text{total}}}(1 - \eta_{\text{fin}}) \ . \end{aligned} \tag{47}$$

The **fin effectiveness** ϵ_{fin} is defined as the ratio of Q_{fin} to the amount of heat that would be dissipated by the fin's "footprint" (A) area on the object if the fin were not there:

$$\epsilon_{\text{fin}} \equiv \frac{Q_{\text{fin}}}{hA(T_b - T_\infty)} \ . \tag{48}$$

Clearly ϵ_{fin} should be greater than 1, or else one would be better able to dissipate heat by leaving the area A of the object exposed instead of installing a fin of the same cross-sectional area protruding from that region of the object.

For simplicity it has been assumed that the cross-sectional area and perimeter of the fin are uniform along the fin's length; for equations applicable to the more general case in which these assumptions are not valid, see [1-3].

2 Forced Convection

Heat flow in a fluid may involve a mechanism unavailable to solids, namely motion of the fluid, or **convection**. In **forced convection**, which will be examined in this section, the fluid is forced to move by some agency other than the heat transfer process itself. (Section 3 will treat **natural convection**, in which the fluid's motion is due to the heat transfer.) Forced convection situations may be classified as external flow over objects or internal flow through hollow objects.

2.1 External Flow

Consider the case of forced convection involving external fluid flow over a flat plate, as shown in Fig. 7. The plate extends from $x = 0$ to $x = L$ and has a width Δz; the y axis extends perpendicularly outward from the plate's surface. Far from the plate, the fluid has a temperature T_∞ and a velocity v_∞ parallel to the plate. The plate has a uniform surface temperature T_s.

As explained in the fluid mechanics summary, the fluid will form a hydrodynamic boundary layer of thickness $\delta_h(x)$ along the surface of the plate. In a similar manner, a thermal boundary layer of thickness $\delta_t(x)$ will also develop.

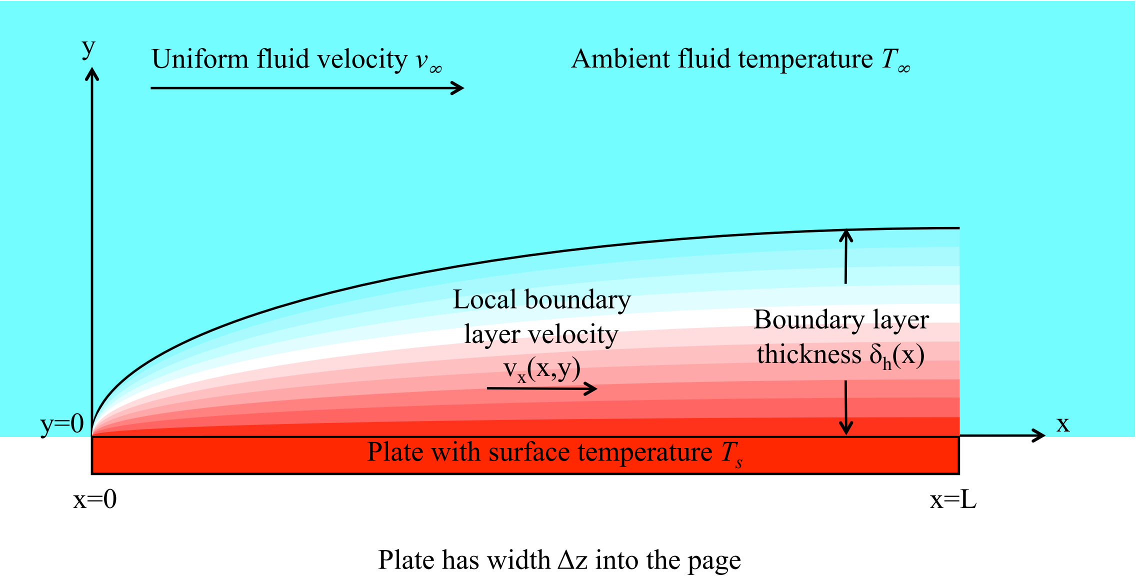


Figure 7. Forced external convection over a flat plate. A hydrodynamic boundary layer of thickness $\delta_h(x)$ and a thermal boundary layer of thickness $\delta_t(x)$ form at the surface of the plate. (Only the hydrodynamic boundary layer is shown for simplicity.) At a certain distance along the plate (determined by the Reynolds number), the boundary layer changes from laminar to turbulent behavior.

If κ is the thermal conductivity of the fluid, the heat flux from the surface may be estimated as

$$q(x) \approx \kappa \frac{T_w - T_\infty}{\delta_t(x)} . \tag{49}$$

The whole problem reduces to finding the correct boundary layer thickness $\delta(x)$ to use in Eq. (49). We'll fill a couple of pages with scary math doing that.

But let's procrastinate for a moment by just concocting alternate definitions of Eq. (49). Equation (49) may be rewritten in the form of Newton's law of cooling with a heat transfer coefficient $h(x)$:

$$q(x) \;=\; h(x)(T_w - T_\infty) \qquad\qquad h(x) \;\equiv\; \frac{\kappa}{\delta_t(x)}\ . \tag{50}$$

As another alternative, zealous heat transferists love to express convective heat transfer in terms of a dimensionless **Nusselt number** Nu:

$$\begin{aligned} q(x) &\approx \kappa\frac{T_w - T_\infty}{x}\,\mathrm{Nu}_x\ , \quad \text{where} & (51)\\ \mathrm{Nu}_x &\equiv \frac{x}{\delta_t(x)} = \frac{h(x)x}{\kappa} & (52)\end{aligned}$$

Okay, enough procrastination; let's get on with the scary math. In order to find the thermal boundary layer thickness $\delta_t(x)$, consider the thermal energy flux $\rho c_p(T - T_\infty)v_x$ flowing in the x direction within the boundary layer, where ρ is the fluid density, c_p is the specific heat capacity of the fluid at constant pressure, and the temperature is measured relative to the temperature T_∞ of the fluid beyond the boundary layer. By integrating this thermal energy flux over the y (vertical) and z (perpendicular to the page in Fig. 7) dimensions of the boundary layer, one obtains the total current of thermal energy flowing within the boundary layer parallel to the wall. The heat flux $q(x)$ coming out of the wall at point x goes into increasing the thermal current within the boundary layer:

$$q(x)\ \Delta z \;=\; \frac{d}{dx}\left[\int_0^{\delta_t} dy\ \rho\ c_p\ (T - T_\infty)\ v_x\ \Delta z\right]\ . \tag{53}$$

For simplicity assume that the temperature and velocity profiles in the boundary layer are linear:

$$\begin{aligned} T - T_\infty &\approx (T_w - T_\infty)\left(1 - \frac{y}{\delta_t}\right)\ ; & (54)\\ v_x &\approx v_\infty\frac{y}{\delta_h}\ . & (55)\end{aligned}$$

[Actually, by using Eq. (49), it has already been implicitly assumed that the temperature profile in the boundary layer is linear.]

Substituting the right-hand sides of Eqs. (49), (54), and (55) into Eq. (53) and using the thermal diffusivity $\alpha \equiv \kappa/\rho c_p$, one finds

$$\begin{aligned} \frac{\alpha}{v_\infty} &\approx \frac{\delta_t}{6}\frac{d}{dx}\left(\frac{\delta_t^2}{\delta_h}\right) \approx \frac{\delta_t}{6}\left(2\frac{\delta_t}{\delta_h}\frac{d\delta_t}{dx} - \frac{\delta_t^2}{\delta_h^2}\frac{d\delta_h}{dx}\right)\\ &\approx \frac{\delta_t}{6}\left(\frac{\delta_t}{\delta_h}\frac{d\delta_t}{dx}\right)\ . \end{aligned} \tag{56}$$

On the last line of Eq. (56), it has been assumed that the hydrodynamic and thermal boundary layer thicknesses are linearly proportional to each other, so that $d\delta_h/dx = (d\delta_h/d\delta_t)d\delta_t/dx = (\delta_h/\delta_t)d\delta_t/dx$. The validity of this assumption will be demonstrated shortly.

Integrating Eq. (56) and assuming that $\delta_t = 0$ at $x = 0$, it is found that

$$\delta_t \;\approx\; \left(\frac{18\delta_h \alpha x}{v_\infty}\right)^{1/3} \; . \qquad (57)$$

As derived in *Fluid Mechanics and Aerodynamics* ?.?, if the fluid's kinematic viscosity is $\nu = \mu/\rho$ [m^2/sec], where μ is the viscosity [kg/s·m], the hydrodynamic boundary layer thickness will be

$$\delta_h \;\approx\; 5\sqrt{\frac{\nu x}{v_\infty}} \;=\; \frac{5x}{\sqrt{\mathrm{Re}_x}} \;, \qquad \text{in which the \textbf{Reynolds number} is } \mathrm{Re}_x \equiv \frac{v_\infty x}{\nu} \qquad (58)$$

or equivalently, $x/v_\infty \;\approx\; \delta_h^2/25\nu$. The Reynolds number is a measure of how turbulent a fluid flow is; flow is laminar for small Reynolds numbers and turbulent for large ones, as explained in *Fluid Mechanics and Aerodynamics* ?.?. Substituting this result into Eq. (57) and approximating $(18/25)^{1/3} \approx 0.9$ as 1 for simplicity, one finds

$$\delta_t \;\approx\; \mathrm{Pr}^{-1/3}\delta_h \qquad (59)$$

where the **Prandtl number** is

$$\mathrm{Pr} \;\equiv\; \frac{\nu}{\alpha} \;=\; \frac{\nu}{\kappa/(\rho c_p)} \;=\; \frac{c_p \mu}{\kappa} \qquad (60)$$

Thus the thermal and hydrodynamic boundary layer widths are proportional to each other as was assumed earlier. The Prandtl number of the fluid may be regarded as the ratio of the diffusion constant for momentum (the kinematic viscosity) to the diffusion constant for thermal energy. For air and most other fluids of interest, the thermal and hydrodynamic boundary layer widths are quite comparable to each other, or Pr ≈ 1. Nonetheless, we'll keep including the Prandtl number just to keep everything as broadly applicable as possible.

Using Eqs. (59) and (58), Eq. (49) for the heat flux may be rewritten as

$$\begin{aligned} q(x) \;&=\; \frac{\mathrm{Pr}^{1/3}\sqrt{v_\infty}}{5\sqrt{\nu}}\,\frac{\kappa(T_s - T_\infty)}{\sqrt{x}} \\ &=\; h(x)(T_s - T_\infty) \qquad \text{with } \; h(x) \;\equiv\; 0.2\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\,\kappa/x \qquad (61)\end{aligned}$$

A more rigorous version of this derivation would employ more complicated temperature and velocity profiles than those in Eqs. (54) and (55), but it would only find a different numerical constant [1-3]:

$$h(x) \;=\; 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\,\frac{\kappa}{x} \qquad (62)$$

The total heat transfer from the entire surface may be obtained from an averaged heat transfer coefficient $\overline{h_L}$, so that $Q = \overline{h_L}A(T_s - T_\infty)$ for a surface of area A, where

$$\overline{h_L} \;\equiv\; \frac{1}{L}\int_0^L h(x)dx \;=\; 2h(x=L) \;=\; 0.664\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2}\,\frac{\kappa}{L} \; . \qquad (63)$$

Converting to those silly Nusselt numbers, h and $\overline{h_L}$ become:

$$\left.\begin{aligned} \mathrm{Nu}_x \;&\equiv\; \tfrac{h(x)\,x}{\kappa} \;=\; 0.332\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3} \\ \overline{\mathrm{Nu}_L} \;&\equiv\; \tfrac{\overline{h_L}L}{\kappa} \;=\; 0.664\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3} \end{aligned}\right\} \;\text{laminar flow over flat plate with uniform } T_s \qquad (64)$$

The boundary layer becomes turbulent for $\mathrm{Re}_x \geq \mathrm{Re}_{\mathrm{transition}} \approx 5 \cdot 10^5$ and the power dependence of the heat transfer on Re_x changes. In this case, the $\overline{\mathrm{Nu}_L}$ must account for the laminar boundary layer heat transfer at small x and the turbulent boundary layer heat transfer at large x [2, 3]:

$$\left.\begin{array}{lcl} \mathrm{Nu}_x & \approx & 0.03\,\mathrm{Re}_x^{4/5}\mathrm{Pr}^{1/3} \\ \overline{\mathrm{Nu}_L} & \approx & 0.037\,\mathrm{Pr}^{1/3}\left[\mathrm{Re}_L^{4/5} - \left(\mathrm{Re}_{\mathrm{transition}}^{4/5} - 17.95\mathrm{Re}_{\mathrm{transition}}^{1/2}\right)\right] \end{array}\right\} \begin{array}{l}\text{turbulent flow over flat} \\ \text{plate with uniform } T_s\end{array} \qquad (65)$$

The precise numerical constants can vary somewhat if it is the surface heat flux which is uniform instead of the surface temperature [1-3], if the surface is not perfectly smooth, etc.

These results may be applied to external flow over objects other than flat plates by modifying the Nusselt number from Eq. (64) to include empirical fudge factors:

$$\overline{\mathrm{Nu}_L} \;=\; C_1 \;+\; C_2\, f_1(\mathrm{Re}_L)\, f_2(\mathrm{Pr})\, \mathrm{Re}_L^{1/2}\, \mathrm{Pr}^{1/3}\,, \qquad (66)$$

in which C_1 and C_2 are constants of order unity, $f_1(\mathrm{Re}_L)$ and $f_2(\mathrm{Pr})$ are relatively weak functions, and L is the object's characteristic length. For cylinders and spheres, $L = D$. The modifications of the Nusselt number are due to the fact that the boundary layer may be distorted as it moves around the object and may even separate from the object's surface. See [1-3] for more details.

An important application of forced external convection is heat transfer to vehicles reentering the atmosphere from space [6]. For simplicity consider a spherical vehicle of radius R in laminar flow. By simply adjusting the constant from Eq. (64) to account for the spherical geometry, one can use $\overline{\mathrm{Nu}_L} \approx 2\mathrm{Re}_R^{1/2}\mathrm{Pr}^{1/3}$, or

$$q \;\approx\; 2\,\mathrm{Re}_R^{1/2}\,\mathrm{Pr}^{1/3}\,\frac{\kappa(T_\infty - T_s)}{R} \qquad (67)$$

For air, the Prandtl number is $\mathrm{Pr} \approx 1$ and may be ignored. At the high velocities typical of reentry, the kinetic energy density of the flow $\rho v^2/2$ is converted to thermal energy density $\rho c_p T_\infty$, producing a temperature $T_\infty \approx v^2/(2c_p)$ just beyond the boundary layer that is much higher than the surface temperature of the vehicle, $T_\infty \gg T_s$. Using these approximations and the Reynolds number $\mathrm{Re}_R = \rho v R/\mu$, Eq. (67) becomes

$$q \;\approx\; \frac{\kappa/c_p}{\sqrt{\mu}}\sqrt{\frac{\rho}{R}}\,v^{2.5} \;\approx\; \sqrt{\mu}\sqrt{\frac{\rho}{R}}\,v^{2.5} \qquad (68)$$

in which Eq. (68) used the relation $\mu \approx \kappa/c_p$ from $\mathrm{Pr} = c_p\mu/\kappa \approx 1$.

The viscosity of air at high temperatures (or equivalently at high flow velocities) varies according to Sutherland's law:

$$\mu \;\approx\; 1.5 \times 10^{-6}\sqrt{T_{^\circ K}}\;\frac{\mathrm{kg}}{\mathrm{m\ sec}} \;\approx\; 4 \times 10^{-8}\,v_{\mathrm{m/sec}}\;\frac{\mathrm{kg}}{\mathrm{m\ sec}} \qquad (69)$$

Combining Eqs. (68) and (69), the heat transfer becomes

$$q \;\approx\; 2 \times 10^{-4}\sqrt{\frac{\rho}{R}}\,v^3\;\frac{\mathrm{W}}{\mathrm{m}^2} \qquad (70)$$

Of course, a reentering vehicle is generally not spherical, and the precise local and total heat transfer rates will depend on the specific shape of the vehicle. However, Eq. (70) is extremely useful for obtaining initial estimates of reentry heat transfer before doing more detailed calculations. For more information on reentry and the implications of Eq. (70), see *Fluid Mechanics and Aerodynamics* 3.10 and [6].

2.2 Internal Flow

Consider internal fluid flow in a circular pipe of diameter $D = 2R$. An average fluid velocity may be defined from the mass flow rate, $v_{\text{avg}} \equiv \dot{m}/(\pi R^2 \rho)$. The Reynolds number of the flow is then:

$$\text{Re}_D \equiv \frac{v_{\text{avg}}\, D}{\nu} = \frac{4\dot{m}}{\pi \mu D}\,, \tag{71}$$

As explained in *Fluid Mechanics and Aerodynamics* ?.?, the flow is laminar for $\text{Re}_D < 2300$ and turbulent for $\text{Re}_D > 2300$. Laminar flow is hydrodynamically fully developed after a hydrodynamic entry length 0.05 $\text{Re}_D\ D$ and thermally fully developed after a thermal entry length 0.017 $\text{Re}_D\ \text{Pr}\ D$ [1-3]. (Hydrodynamic and thermal entry lengths are not very well defined for turbulent flow.)

For a cylindrically symmetric system with the z axis parallel to the pipe, $v_r = 0$, and $\partial/\partial t = 0$, Eq. (26) becomes:

$$v_z \frac{\partial T}{\partial z} = \alpha \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)\,. \tag{72}$$

It is conventional to define an average **bulk temperature** T_b of the fluid as:

$$T_b \equiv \frac{\int_0^R (\rho v_z c_p)(T)(2\pi r\ dr)}{\int_0^R (\rho v_z c_p)(2\pi r\ dr)}\,. \tag{73}$$

For fully developed flow with $q_w \equiv q(r = R) = \text{constant}$, $T_w \equiv T(R)$ and T_b will increase linearly with z at the same rate. The thermal energy balance, $2\pi R q_w dz = v_{\text{avg}}(\pi R^2)\rho c_p dT_b$, means that

$$\frac{\partial T_b}{\partial z} = \frac{\partial T_w}{\partial z} = \frac{\partial T}{\partial z} = \frac{2 q_w \alpha}{v_{\text{avg}} R \kappa}\,. \tag{74}$$

As derived in *Fluid Mechanics and Aerodynamics* ?.?, the velocity profile for fully developed laminar flow is $v_z = 2v_{\text{avg}}[1 - (r/R)^2]$. By inserting this profile and the result of Eq. (74) into Eq. (72), one finds

$$4\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{q_w}{R\kappa} = \alpha \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\,. \tag{75}$$

Integrating Eq. (75) twice, one obtains $T(r)$ for hydrodynamically and thermally developed flow:

$$T = \frac{q_w R}{\kappa}\left[\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right] - \frac{7}{24}\frac{q_w R}{\kappa} + T_b\,. \tag{76}$$

The last two terms on the right side of Eq. (76) are additive constants chosen to satisfy Eq. (73).

Evaluating Eq. (76) at $r = R$ yields the temperature difference,

$$T_w - T_b = \frac{11}{24}\frac{q_w R}{\kappa} = \frac{11}{48}\frac{q_w D}{\kappa}\,. \tag{77}$$

Therefore the Nusselt number for fully developed laminar flow with constant q_w is:

$$\overline{\mathrm{Nu}_D} \equiv \frac{\overline{h}D}{\kappa} \equiv \frac{q_w D}{(T_w - T_b)\kappa} = \frac{48}{11} \approx 4.36\ . \tag{78}$$

For fully developed laminar flow with constant T_w instead of constant q_w, the Nusselt number is only slightly different [1-3]:

$$\overline{\mathrm{Nu}_D} \equiv \frac{\overline{h}D}{\kappa} \approx 3.66 \tag{79}$$

For fully developed turbulent flow with constant T_w or constant q_w, an empirical correlation is [3]:

$$\overline{\mathrm{Nu}_D} \equiv \frac{\overline{h}D}{\kappa} \approx \frac{(f/8)(\mathrm{Re}_D - 1000)\mathrm{Pr}}{1 + 12.7(f/8)^{1/2}(\mathrm{Pr}^{2/3} - 1)}\ , \tag{80}$$

in which $f \equiv (0.79 \ln \mathrm{Re}_D - 1.64)^{-2}$ and all properties are evaluated at T_b.

For non-circular pipes with cross-sectional area A_c and perimeter P, use $D \equiv 4A_c/P$.

2.3 Heat Exchangers

An important application of forced convective heat transfer is heat exchangers, which are designed to transfer heat from one fluid to another.

A simple example of a heat exchanger is a circular pipe which has one fluid flowing through it and another fluid flowing around it. If the pipe has inner diameter D_i, outer diameter D_o, length L, and thermal conductivity κ, the total thermal resistance between the two fluids may be written as

$$\begin{aligned}\frac{1}{UA} &\equiv \frac{1}{U_i A_i} = \frac{1}{U_o A_o} \\ &= \frac{1}{h_i A_i} + \frac{1}{h_{\text{fouling},\,i} A_i} + \frac{\ln(D_o/D_i)}{2\pi L \kappa} + \frac{1}{h_{\text{fouling},\,o} A_o} + \frac{1}{h_o A_o}\ , \end{aligned} \tag{81}$$

in which $A_i \equiv \pi D_i L$ and $A_o \equiv \pi D_o L$. Note that convection, conduction, and fouling resistances have been included. The overall heat transfer coefficient U is basically just a net "h" for the system; heat transfer gurus traditionally use a different letter just to sound more impressive and confusing.

To make all of this look even more impressive and confusing, Eq. (81) may be generalized for the thermal resistance between the hot (subscript h) and cold (subscript c) sides of a heat exchanger which may have fins and a geometry other than that of a circular pipe:

$$\begin{aligned}\frac{1}{UA} &\equiv \frac{1}{U_c A_c} = \frac{1}{U_h A_h} \\ &= \frac{1}{\eta_{0,c} h_c A_c} + \frac{1}{\eta_{0,c} h_{\text{fouling},\,c} A_c} + R_{\text{wall}} + \frac{1}{\eta_{0,h} h_{\text{fouling},\,h} A_h} + \frac{1}{\eta_{0,h} h_h A_h} \end{aligned} \tag{82}$$

Consider a parallel flow heat exchanger, as shown in Fig. 8(a); the fluids on each side of the heat exchanger's wall flow in the same direction. For the moment assume that the temperature T_h of the hotter fluid is constant all along the length of the heat exchanger, so that that fluid's temperature at the heat exchanger's inlet is the same as at the heat exchanger's outlet, $T_{h,i} = T_{h,o}$. If the colder fluid has mass flow rate $\dot{m}_c$, specific heat $c_{p,c}$, inlet temperature $T_{c,i}$, and outlet temperature $T_{c,o}$, the heat transfer between the two fluids may be written as

$$Q = \dot{m}_c\, c_{p,c}\, (\Delta T_i - \Delta T_o)\ , \tag{83}$$

in which $\Delta T_i \equiv T_{h,i} - T_{c,i}$ is the temperature difference between the fluids at the inlet and $\Delta T_o \equiv T_{h,o} - T_{c,o}$ is the temperature difference at the outlet.

The temperature difference between the two fluids declines exponentially with position along the heat exchanger, and the equation for this exponential decline may be turned around to provide an expression for $\dot{m}_c\, c_{p,c}$:

$$\Delta T_o = \Delta T_i \exp\left(-\frac{UA}{\dot{m}_c\, c_{p,c}}\right) \qquad \rightarrow \qquad \dot{m}_c\, c_{p,c} = -\frac{UA}{\ln(\Delta T_o/\Delta T_i)}\ . \tag{84}$$

Substituting the expression for $\dot{m}_c\, c_{p,c}$ from Eq. (84) into Eq. (83), one finds:

$$Q = UA(\Delta T)_{lm}\ , \tag{85}$$

in which the log mean temperature difference $(\Delta T)_{lm}$ has been defined as:

$$(\Delta T)_{lm} \equiv \frac{\Delta T_o - \Delta T_i}{\ln(\Delta T_o/\Delta T_i)} = \frac{(T_{h,i} - T_{c,i}) - (T_{h,o} - T_{c,o})}{\ln[(T_{h,i} - T_{c,i})/(T_{h,o} - T_{c,o})]} \quad \text{for parallel flow heat exchangers} \tag{86}$$

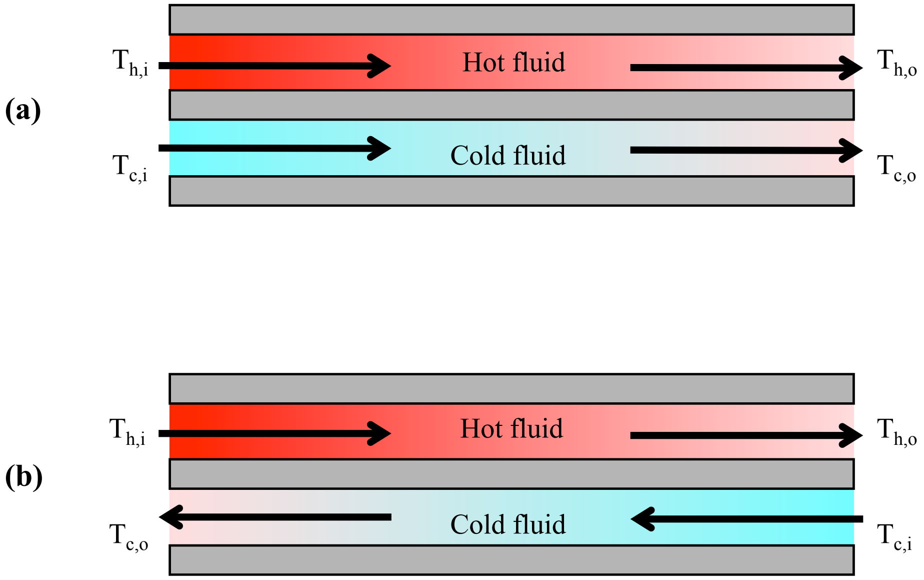


Figure 8. Schematic illustrations of heat exchangers, showing (a) parallel flow and (b) counterflow configurations. The diagram shows the inlet and outlet temperatures for both the hot and cold fluids.

Although Eqs. (85) and (86) were derived using the assumption that $T_{h,i} = T_{h,o}$, they are also applicable to cases in which $T_{h,i} \neq T_{h,o}$, as may be shown with a bunch more algebra [1-3].

For counterflow heat exchangers, in which the hot and cold fluids flow through the heat exchanger in opposite directions as shown in Fig. 8(b), Eq. (85) is still valid provided that one uses an appropriate expression for the log mean temperature difference. Since the inlet-end of the heat exchanger for one fluid is now the outlet-end for the other fluid, it is necessary to switch the inlet and outlet temperatures for one of the fluids. Rewriting Eq. (86) with the interchange of variables $T_{c,i} \leftrightarrow T_{c,o}$, the log mean temperature difference for counterflow heat exchangers is found to be:

$$(\Delta T_{lm}) \equiv \frac{(T_{h,i} - T_{c,o}) - (T_{h,o} - T_{c,i})}{\ln[(T_{h,i} - T_{c,o})/(T_{h,o} - T_{c,i})]} \qquad \text{for counterflow heat exchangers} \tag{87}$$

For a counterflow heat exchanger with $(\dot{m}c_p)_c = (\dot{m}c_p)_h$, the temperature difference $T_h - T_c$ remains constant along the length of the heat exchanger.

For any type of heat exchanger, the total heat transfer between the two fluids may be written as

$$Q = \dot{m}_h c_{p,h}(T_{h,i} - T_{h,o}) = \dot{m}_c c_{p,c}(T_{c,o} - T_{c,i}) \tag{88}$$

If $(\dot{m}c_p)_a \gg (\dot{m}c_p)_b$ or if fluid "a" is undergoing a phase change in the heat exchanger, fluid "a" may be treated as approximately isothermal, so that $T_{a,i} \approx T_{a,o}$.

Define $(\dot{m}c_p)_{\min}$ to be the smaller of the two quantities $(\dot{m}c_p)_h$ and $(\dot{m}c_p)_c$, and likewise define $(\dot{m}c_p)_{\max}$ to be the larger of those two quantities. The fluid which has the smaller value of $(\dot{m}c_p)$ will more rapidly approach the temperature of the other fluid. Once the fluid with the smaller $(\dot{m}c_p)$ reaches the temperature of the other fluid, no more heat transfer can take place. Thus the maximum heat transfer which can possibly occur is determined by $(\dot{m}c_p)_{\min}$:

$$Q_{\max} \;\equiv\; (\dot{m}c_p)_{\min}(T_{h,i} - T_{c,i}) \tag{89}$$

Because the temperature of the fluid with the larger $(\dot{m}c_p)$ also approaches the temperature of the fluid with the smaller $(\dot{m}c_p)$, the actual heat transfer is less than the upper bound of Eq. (89).

A definition which will be useful in a moment is the **number of transfer units (NTU)**:

$$\mathrm{NTU} \;\equiv\; \frac{UA}{(\dot{m}c_p)_{\min}}\;. \tag{90}$$

Another useful quantity is the ratio of the heat capacities:

$$C_r \;\equiv\; \frac{(\dot{m}c_p)_{\min}}{(\dot{m}c_p)_{\max}}\;. \tag{91}$$

The effectiveness ϵ of a heat exchanger is defined as

$$\epsilon \;\equiv\; Q/Q_{\max} \;=\; \mathrm{fn}\,(\mathrm{NTU},\; C_r,\; \mathrm{geometry})\;. \tag{92}$$

Note that the effectiveness is only a function of NTU, C_r, and the geometry or type of heat exchanger (parallel flow, counterflow, multipass, cross-flow, etc.).

By mucking through a fair amount of algebra and rewriting the earlier equations for heat exchangers, the effectiveness relations for parallel flow and counterflow heat exchangers are found to be [2]:

$$\epsilon \;=\; \frac{1 - \exp[-(1 + C_r)\mathrm{NTU}]}{1 + C_r} \qquad \text{for single-pass parallel flow heat exchangers} \tag{93}$$

$$\epsilon \;=\; \frac{1 - \exp[-(1 - C_r)\mathrm{NTU}]}{1 - C_r\exp[-(1 - C_r)\mathrm{NTU}]} \qquad \text{for single-pass counterflow heat exchangers} \tag{94}$$

Effectiveness relations for other types of heat exchangers may be found in [1-3]. With this effectiveness-NTU method, the prerequisite knowledge needed for heat exchanger calculations includes the heat capacities of the fluids instead of the outlet temperatures, as was the case for the log mean temperature difference technique. This is usually an advantage.

3 Natural Convection

In **natural** (or **free**) **convection**, the fluid's velocity is supplied entirely by the thermal effects. Simple analytical models of heat transfer during natural convection may be constructed for certain cases, although for more complicated situations it is necessary to rely upon experimentally determined empirical correlations. Fortunately, the physical principles of natural convection and the fundamental functional dependences of the heat transfer coefficient may be ascertained from the analytical methods.

3.1 Natural Convection Along a Vertical Wall

As a simple example of natural convection, consider the situation shown in Fig. 9. A generally motionless fluid at bulk temperature T_∞ is in contact with a vertical heated wall of temperature T_w. Fluid immediately adjacent to the wall is heated and expands, then rises along the wall's surface due to the buoyant effect of its lower density. This rising fluid forms a boundary layer between the wall and the bulk of the fluid and hence affects the rate of heat transfer from the wall to the fluid.

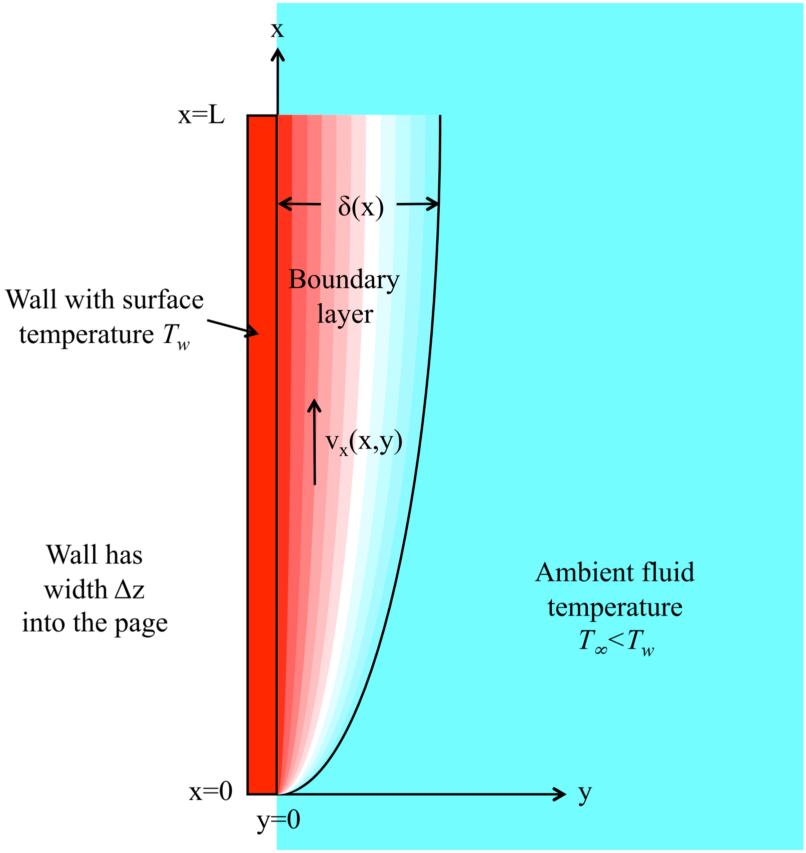


Figure 9. Natural convection along a heated vertical wall. The hydrodynamic and thermal boundary layers are assumed to have the same width $\delta(x)$.

As shown in Fig. 9, the x-axis may be chosen to run vertically along the wall, so that $x = 0$ at the bottom and $x = L$ at the top. The y-axis protrudes perpendicularly from the wall. It will be assumed that the width of the wall in the remaining dimension is Δz and that spatial variations in this direction may be neglected. As an added simplification, it will be assumed that the fluid and thermal boundary layers have the same thickness $\delta(x)$ in the y-direction.

If κ is the fluid's thermal conductivity, the heat flux coming out of the wall may be estimated as

$$q(x) \;\approx\; \kappa \frac{T_w - T_\infty}{\delta(x)} \; . \tag{95}$$

Thus in order to find the heat flux, it is necessary to calculate the boundary layer's thickness. To do this, one may consider the amount of kinetic energy per unit height contained within the boundary layer. This energy changes as the fluid moves vertically and work is done on it by forces. Since the work input is simply the sum of the forces integrated over the vertical distance, one finds that

$$\begin{aligned}
\frac{d}{dx}\left(\begin{array}{c}\text{kinetic energy}\\ \text{per unit height}\end{array}\right) &= \left(\begin{array}{c}\text{buoyancy force}\\ \text{per unit height}\end{array}\right) - \left(\begin{array}{c}\text{skin friction force}\\ \text{per unit height}\end{array}\right)\\
\frac{d}{dx}\int_0^{\delta(x)} dy\; \rho(y)\; v_x^2(x,y)\; \Delta z &= \int_0^{\delta(x)} dy\; g\; [\rho_\infty - \rho(y)]\; \Delta z \;-\; \nu\; \rho(y=0)\; \left[\frac{dv_x}{dy}\right]_{y=0} \Delta z\\
\frac{d}{dx}\Big[\delta(x)\; \rho(y=0)\; v_x^2(x)\Big] &\approx \delta(x)\; g\; [\rho_\infty - \rho(y=0)] \;-\; \nu\; \rho(y=0)\; \left[\frac{v_x(x)}{\delta(x)}\right] .
\end{aligned} \tag{96}$$

$\rho(y)$ is the fluid density at distance y from the wall, ρ_∞ is the density far from the wall, ν is the kinematic viscosity, and $g \approx 9.807$ m/sec^2 is the gravitational acceleration. The approximation that has been made in Eq. (96) assumes that the boundary layer's properties may be essentially averaged over the layer's width δ in the y direction so that only x-dependences remain.

Modeling the fluid as an ideal gas with constant pressure, one finds from the ideal gas law that $\rho \propto 1/T$. Therefore $\Delta\rho/\rho \;=\; -\Delta T/T$, so the buoyancy term may be rewritten in terms of the temperature difference across the boundary layer. With this change, Eq. (96) becomes

$$\frac{d}{dx}\Big[\delta(x)\; v_x^2(x)\Big] \;\approx\; \delta(x)\; g\; \left(\frac{T_w - T_\infty}{T_{\text{avg}}}\right) \;-\; \nu \frac{v_x(x)}{\delta(x)} \; , \tag{97}$$

where $T_{\text{avg}} \equiv (T_w + T_\infty)/2$.

In order to solve for $\delta(x)$, it is necessary to eliminate the pesky unknown $v_x(x)$. As a very crude approximation, one could assume that the boundary layer looks the same all the way up, so $d/dx = 0$ in Eq. (97) and the buoyancy and friction terms on the right-hand-side of Eq. (97) cancel each other out. This approximation yields a boundary layer "terminal velocity" of $v_x \approx g(T_w - T_\infty)\delta^2/\nu T_{\text{avg}}$. In reality these terms do not exactly cancel, and the boundary layer width and velocity vary with height x, so one must include a fudge factor C_1 of order unity:

$$v_x(x) \;\approx\; C_1\, \frac{g\; (T_w - T_\infty)\; [\delta(x)]^2}{\nu\; T_{\text{avg}}} \; . \tag{98}$$

Substituting this expression for $v_x(x)$ into Eq. (97) and solving for $\delta(x)$, it is found that

$$\begin{aligned} \delta(x) &\approx C_2 \left[\frac{1}{g(T_w - T_\infty)/T_\text{avg}\nu^2}\, x \right]^{1/4}, \quad \text{with a redesigned fudge factor } C_2 \equiv \left[\frac{4(1-C_1)}{5C_1^2} \right]^{1/4} \\ &\approx C_2\, \mathrm{Gr}_x^{-1/4}\, x\,, \qquad \text{where the \textbf{Grashof number} is } \mathrm{Gr}_x \equiv \frac{g(T_w - T_\infty)x^3}{\nu^2 T_\text{avg}} \end{aligned} \tag{99}$$

Inserting this expression for $\delta(x)$ into Eq. (95), the heat flux may be written as

$$q(x) \equiv h(x)(T_w - T_\infty) = \frac{\mathrm{Gr}_x^{1/4}\, \kappa\, (T_w - T_\infty)}{C_2\, x}\,, \tag{100}$$

so the Nusselt number (basically the dimensionless heat transfer) is

$$\mathrm{Nu}_x \equiv \frac{h(x)\, x}{\kappa} = \frac{1}{C_2} \mathrm{Gr}_x^{1/4}\,. \tag{101}$$

Thus the heat transfer and Nusselt number are proportional to the fourth root of the Grashof number. The Grashof number may be thought of as the ratio of the buoyancy force to the viscous force acting on the boundary layer.

A more rigorous version of this derivation [2] confirms this dependence on the Grashof number, provides the fudge factor C_2, and also indicates that the heat transfer expression must be corrected to account for variations of the Prandtl number $\mathrm{Pr} \equiv \nu/\alpha$. (The inclusion of the Prandtl number helps weigh the relative importance of fluid boundary layer effects and thermal boundary layer effects.) The results of the rigorous derivation are:

$$\begin{aligned} \mathrm{Nu}_x &= 0.508 \left(\frac{\mathrm{Pr}^2}{0.952 + \mathrm{Pr}} \right)^{1/4} \mathrm{Gr}_x^{1/4} \\ &= 0.508 \left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}} \right)^{1/4} \mathrm{Ra}_x^{1/4}, \qquad \text{where the \textbf{Rayleigh number} is } \mathrm{Ra}_x \equiv \mathrm{Pr}\, \mathrm{Gr}_x \\ \overline{\mathrm{Nu}_L} &= 0.678 \left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}} \right)^{1/4} \mathrm{Ra}_L^{1/4}\,. \end{aligned} \tag{102}$$

The Rayleigh number is the ratio of buoyancy and viscosity effects. If the convective boundary layer is far more buoyant than viscous (large Rayleigh number), it will rise rapidly up the wall and be rapidly replaced by fresh fluid, and the heat transfer (Nusselt number) will be large. If the convective boundary layer is less buoyant and more viscous (small Rayleigh number), it will cling to the wall and help to insulate it, and the heat transfer (Nusselt number) will be small.

Typically $\mathrm{Ra}_L > 10^5$. For $\mathrm{Ra}_L < 10^5$ some of the assumptions underlying the modeling of the boundary layer break down, and a better empirical fit is obtained by adding a constant of order unity to the right-hand side of the relations in Eq. (102). See [2, 3] for more details.

For Rayleigh numbers larger than roughly 10^9, the buoyancy force on the boundary layer is so great that the boundary layer becomes hydrodynamically unstable and turbulent. Empirical correlations suitable for this case are given in [3].

Nothing in all of the above equations actually requires the wall to be hotter than the surrounding fluid, so these results may also be applied to boundary layers moving down refrigerated walls instead of boundary layers moving up heated walls.

3.2 Natural Convection in Other Situations

Another important example of natural convection is convective heat transfer involving the upper surface of a horizontal heated plate or the lower surface of a horizontal refrigerated plate. In these cases the boundary layer is unstable and keeps separating from the plate. The heated boundary layer fluid rises away from the hot plate and the cooled boundary layer fluid falls down from the cold plate. The situation created by these thermal effects–a heavy (cold) fluid unstably resting on top of a light (hot) fluid–is known as the **Rayleigh-Taylor instability** [4] and is also encountered in very different settings, such as plasma physics. The unstable, turbulent nature of the boundary layer fluid separating from the surface and being replaced by fresh fluid rushing in makes the problem extremely difficult to model analytically; one is generally forced to resort to empirical correlations. Quite remarkably, the Nusselt number for these cases is still a constant of order unity times the fourth root of the Rayleigh number, just as it was for the vertical plate [3]:

$$\overline{\mathrm{Nu}_L} \;=\; 0.54\mathrm{Ra}_L^{1/4}\,, \qquad\qquad \text{for } \mathrm{Ra}_L < 10^7 \tag{103}$$

in which the characteristic length L of the plate is defined as the plate's surface area divided by its perimeter. At very large Rayleigh numbers ($\mathrm{Ra}_L > 10^7$), the power dependence on the Rayleigh number changes slightly [3]:

$$\overline{\mathrm{Nu}_L} \;=\; 0.15\mathrm{Ra}_L^{1/3}\,, \qquad\qquad \text{for } \mathrm{Ra}_L > 10^7 \tag{104}$$

All fluid properties should be evaluated at an average temperature of $T_{\mathrm{avg}} \equiv (T_w + T_\infty)/2$ unless otherwise noted.

Empirical expressions for natural convection around surfaces of other shapes are given in [2, 3]. Generally these expressions are of the same form as has been discussed,

$$\overline{\mathrm{Nu}_L} \;=\; C_3 \;+\; C_4\, f(\mathrm{Pr})\, \mathrm{Ra}_L^{1/4}\,, \tag{105}$$

in which C_4 is a constant of order unity, $f(\mathrm{Pr})$ is a relatively weak function of the Prandtl number, and C_3 is a constant of order unity which only becomes important at very low Rayleigh numbers.

For cases in which both natural convection and forced convection are present, one must decide if either type of convection may be safely neglected. This determination may be made by comparing the Grashof number (describing the magnitude of natural convection effects) and the Reynolds number (or actually the square of the Reynolds number, describing forced convection effects). If $\mathrm{Gr}_L \ll \mathrm{Re}_L^2$, natural convection may be neglected. For $\mathrm{Gr}_L \gg \mathrm{Re}_L^2$, one may neglect forced convection effects. If $\mathrm{Gr}_L \sim \mathrm{Re}_L^2$, natural and forced convection must be considered simultaneously. For methods of doing this, see [3] and the scientific literature cited therein.

4 Condensation

When a cool surface is surrounded by a warm saturated gaseous vapor, some of the vapor may condense into a liquid on the surface. The condensation process strongly affects the rate of heat transfer from the vapor to the surface.

4.1 Laminar Condensation on a Flat, Vertical Surface

With reference to Fig. 10, consider a cool vertical wall with temperature T_w and height L which is exposed to a saturated vapor with temperature T_{sat} and density ρ_v. On the wall the vapor will condense into a liquid film layer of density ρ_l and thickness $\delta(x)$, in which x is the distance from the top of the wall. When the vapor condenses into a liquid on the wall's surface, it imparts to the wall an energy of h_{fg} Joules per kg of condensed liquid, where h_{fg} is called the **latent heat of vaporization** for the condensing substance. Thus the condensing vapor transfers heat to the wall. For condensation of steam to water, h_{fg} is approximately 2300 kJ/kg, depending on the exact temperature. Because this value is so large, condensation can transfer a large amount of heat energy.

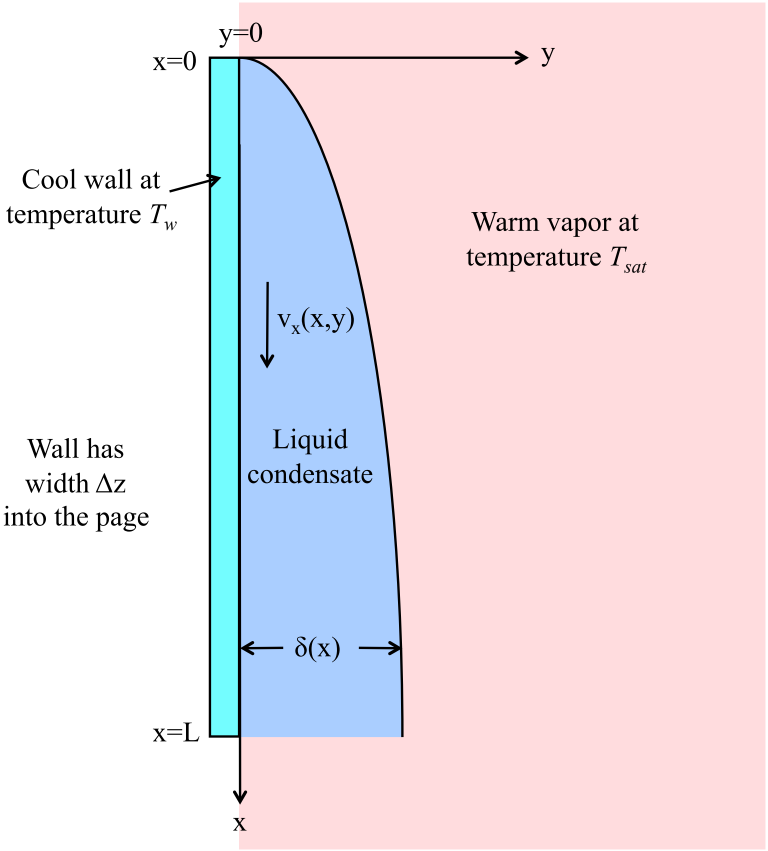


Figure 10. Condensation on a vertical wall. The layer of liquid condensate on the wall has width $\delta(x)$. Initially the liquid flow within the layer is laminar. If the wall is tall enough and the Reynolds number of the liquid condensate layer becomes large enough, the liquid flow within the layer will eventually become first wavy laminar and then turbulent.

In this section, all liquid properties are denoted by the subscript "l" and should be evaluated at a film temperature of $T_f \equiv (T_w + T_{sat})/2$. The subscript "$v$" denotes vapor properties evaluated at temperature T_{sat}. The latent heat h_{fg} should be evaluated at temperature T_{sat}.

If κ_l is the thermal conductivity of the liquid film, the heat flux imparted to the wall at position x may be approximated as:

$$q(x) \approx \kappa_l \frac{T_{sat} - T_w}{\delta(x)} . \qquad (106)$$

Even if the actual vapor temperature far from the wall is greater than T_{sat}, the vapor temperature will drop to T_{sat} at the surface of the liquid film, making Eq. (106) still valid. Therefore one should always use T_{sat} instead of the actual vapor temperature if the vapor is superheated.

Thus, as with forced and natural convection, the entire problem comes down to calculating the correct boundary layer thickness (or in this case film thickness) $\delta(x)$ to use in Eq. (106). To that end, brace yourselves for more scary math.

The Navier-Stokes equation for the film layer may be simplified by assuming that the film is in steady state and is sliding down the wall slowly enough that the inertia terms may be neglected. The pressure gradient inside the film is set equal to the hydrostatic pressure gradient of the surrounding vapor, $dp/dx = \rho_v g$, in which ρ_v is the density of the vapor and $g \approx 9.807$ m/sec^2 is the gravitational acceleration. If y is defined to be the direction perpendicular to the wall and $y = 0$ at the wall's surface, one finds by successively simplifying the Navier-Stokes equation:

$$\begin{aligned} \rho_l \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} &= \rho_l g \hat{\mathbf{x}} - \frac{dp}{dx} \hat{\mathbf{x}} + \mu_l \frac{\partial^2 v_x}{\partial y^2} \hat{\mathbf{x}} \\ 0 &\approx \rho_l g \hat{\mathbf{x}} - \rho_v g \hat{\mathbf{x}} + \mu_l \frac{d^2 v_x}{dy^2} \hat{\mathbf{x}} \\ \rightarrow \frac{d^2 v_x}{dy^2} &\approx -\frac{(\rho_l - \rho_v) g}{\mu_l} . \end{aligned} \qquad (107)$$

μ_l is the viscosity of the liquid condensate. Integrating Eq. (107) and using the boundary conditions $v_x(y = 0) = 0$ at the wall and $[\partial v_x / \partial y]_{y=\delta} = 0$ at the film surface yields a parabolic velocity profile in the fluid layer:

$$v_x(y) = \frac{(\rho_l - \rho_v) g \delta^2}{2\mu_l} \left[2 \left(\frac{y}{\delta} \right) - \left(\frac{y}{\delta} \right)^2 \right] . \qquad (108)$$

Therefore the mass flow rate down the wall at position x (assuming a wall width Δz perpendicular to the page in Fig. 9) is

$$\dot{m}(x) = \int_0^{\delta(x)} \rho_l v_x(y) dy \Delta z = \frac{\rho_l (\rho_l - \rho_v) g \delta^3 \Delta z}{3\mu_l} . \qquad (109)$$

Differentiating Eq. (109), the additional mass flow per unit length due to condensation within the region dx is:

$$\frac{d\dot{m}(x)}{dx} = \frac{\rho_l (\rho_l - \rho_v) g \Delta z \delta^2}{\mu_l} \frac{d\delta}{dx} . \qquad (110)$$

The heat given to the wall at point x by the rate of condensation $d\dot{m}(x)$ within dx of that point is:

$$q(x) = h_{fg} \frac{1}{\Delta z} \frac{d\dot{m}}{dx} = \frac{h_{fg} \rho_l (\rho_l - \rho_v) g \delta^2}{\mu_l} \frac{d\delta}{dx} . \qquad (111)$$

But the heat flow in terms of the film width is also given by Eq. (106). Using that fact, Eqs. (106) and (111) may be combined to eliminate $q(x)$ and produce a differential equation for $\delta(x)$:

$$\frac{\kappa_l \mu_l (T_{sat} - T_w)}{h_{fg} \rho_l (\rho_l - \rho_v) g} \;=\; [\delta(x)]^3 \frac{d\delta}{dx} \; . \tag{112}$$

Integrating Eq. (112) and employing the boundary condition $\delta(x = 0) = 0$, one finds

$$\delta(x) \;=\; \left[\frac{4\kappa_l \mu_l (T_{sat} - T_w) x}{h_{fg} \rho_l (\rho_l - \rho_v) g} \right]^{1/4} \; . \tag{113}$$

By substituting Eq. (113) for the film thickness into Eq. (106), the heat flux is found to be

$$q(x) \;=\; \kappa_l (T_{sat} - T_w) \left[\frac{h_{fg} \rho_l (\rho_l - \rho_v) g}{4\kappa_l \mu_l (T_{sat} - T_w) x} \right]^{1/4} \; . \tag{114}$$

To improve the accuracy of Eq. (114), instead of h_{fg} one should use a corrected value of the latent heat of vaporization, h'_{fg}, which accounts for the heat capacity of the condensing substance [2]:

$$h'_{fg} \;\equiv\; h_{fg} + 0.68 c_{p,l} (T_{sat} - T_w) \;=\; h_{fg} (1 + 0.68\mathrm{Ja}) \; , \tag{115}$$

in which $\mathrm{Ja} \equiv c_{p,l}(T_{sat} - T_w)/h_{fg}$ is defined to be the dimensionless **Jakob number**. Generally this correction is rather minor.

Thus the heat transfer coefficient is:

$$h(x) \;\equiv\; \frac{q(x)}{T_{\mathrm{sat}} - T_w} \;=\; \left[\frac{h'_{fg} \rho_l (\rho_l - \rho_v) g \kappa_l^3}{4(T_{\mathrm{sat}} - T_w) \mu_l x} \right]^{1/4} \; . \tag{116}$$

The heat transfer coefficient averaged over the total height L of the wall is:

$$\begin{aligned} \overline{h_L} \;&\equiv\; \frac{1}{L} \int_0^L h(x) dx \;=\; \frac{4}{3} h(L) \\ &=\; \frac{4}{3} \left[\frac{h'_{fg} \rho_l (\rho_l - \rho_v) g \kappa_l^3}{4(T_{\mathrm{sat}} - T_w) \mu_l L} \right]^{1/4} \qquad (117) \\ &\approx\; \kappa_l \left(\frac{g}{\nu_l^2} \right)^{1/3} \frac{1.47}{\mathrm{Re}_\delta^{1/3}} \qquad \text{for } \mathrm{Re}_\delta < 30 \qquad (118) \end{aligned}$$

in which the Reynolds number Re_δ is defined as

$$\mathrm{Re}_\delta \;\equiv\; \frac{4 v_{\mathrm{avg}} \delta}{\nu_l} \;=\; \frac{4\dot{m}}{\mu_l \, \Delta z} \; , \tag{119}$$

and the condensation rate is

$$\dot{m} \;=\; \frac{Q}{h'_{fg}} \;=\; \frac{\overline{h_L} (L \Delta z)(T_{\mathrm{sat}} - T_w)}{h'_{fg}} \; . \tag{120}$$

4.2 Nonlaminar Condensation on a Flat, Vertical Surface

Calculating the condensation on a vertical surface gets messy when turbulence becomes important, and one must resort to empirical correlations. But at least it is comforting that the laminar case may be solved analytically.

For intermediate Reynolds numbers, $30 < \mathrm{Re}_\delta < 1800$, the film of condensation is somewhat affected by turbulence and is described as "wavy laminar." The average heat transfer coefficient for this case has been found from experimental studies to be [3]:

$$\overline{h_L} \;=\; \kappa_l \left(\frac{g}{\nu_l^2}\right)^{1/3} \frac{\mathrm{Re}_\delta}{1.08\mathrm{Re}_\delta^{1.22} - 5.2} \qquad \text{for } 30 < \mathrm{Re}_\delta < 1800 \tag{121}$$

For large Reynolds numbers, $1800 < \mathrm{Re}_\delta$, the condensate is fully turbulent and the average heat transfer coefficient is given by the empirical relation [3]:

$$\overline{h_L} \;=\; \kappa_l \left(\frac{g}{\nu_l^2}\right)^{1/3} \frac{\mathrm{Re}_\delta}{8750 + 58(\mathrm{Re}_\delta^{3/4} - 253)/\sqrt{\mathrm{Pr}}} \qquad \text{for } 1800 < \mathrm{Re}_\delta \tag{122}$$

The form of Eqs. (121) and (122) may be compared with that of Eq. (118).

4.3 Condensation on Objects with Other Shapes

Condensation on objects of other shapes may be described by equations similar to those derived for laminar condensation on flat vertical surfaces.

The equations for a vertical wall may be applied to a vertical cylinder provided that the maximum film thickness is much less than the circumference of the cylinder [$\delta(L) \ll \pi D$ for cylinder of height L and diameter D].

Furthermore, experimental studies have confirmed that laminar condensation on a horizontal cylinder or a sphere may also be described by Eq. (117) with minor changes to the numerical coefficient. For a horizontal cylinder or a sphere of diameter D, the average heat transfer coefficient is [3]:

$$\overline{h_D} \;=\; C \left[\frac{\rho_l(\rho_l - \rho_v) g h'_{fg} \kappa_l^3}{(T_{\mathrm{sat}} - T_s)\mu_l D}\right]^{1/4} , \tag{123}$$

where the numerical coefficient C is 0.729 for a cylinder and 0.815 for a sphere.

5 Boiling

When a hot surface is immersed in a cooler liquid and begins to boil the liquid, the rate of heat transfer from the surface to the liquid is greatly affected by the boiling process. The exact nature of the boiling process strongly depends on the excess temperature $\Delta T_e \equiv T_s - T_{\text{sat}}$, which is the difference between the temperature of the surface T_s and the saturation temperature T_{sat} of the fluid. As ΔT_e is increased, Fig. 11 shows that a system passes through different behavioral regimes which may be classified as free convection, nucleate boiling, transition boiling, and film boiling. (Even if the liquid temperature far from the surface is less than T_{sat}, the liquid temperature near the surface will reach T_{sat}, so one should always use T_{sat} instead of the actual liquid temperature, if different.)

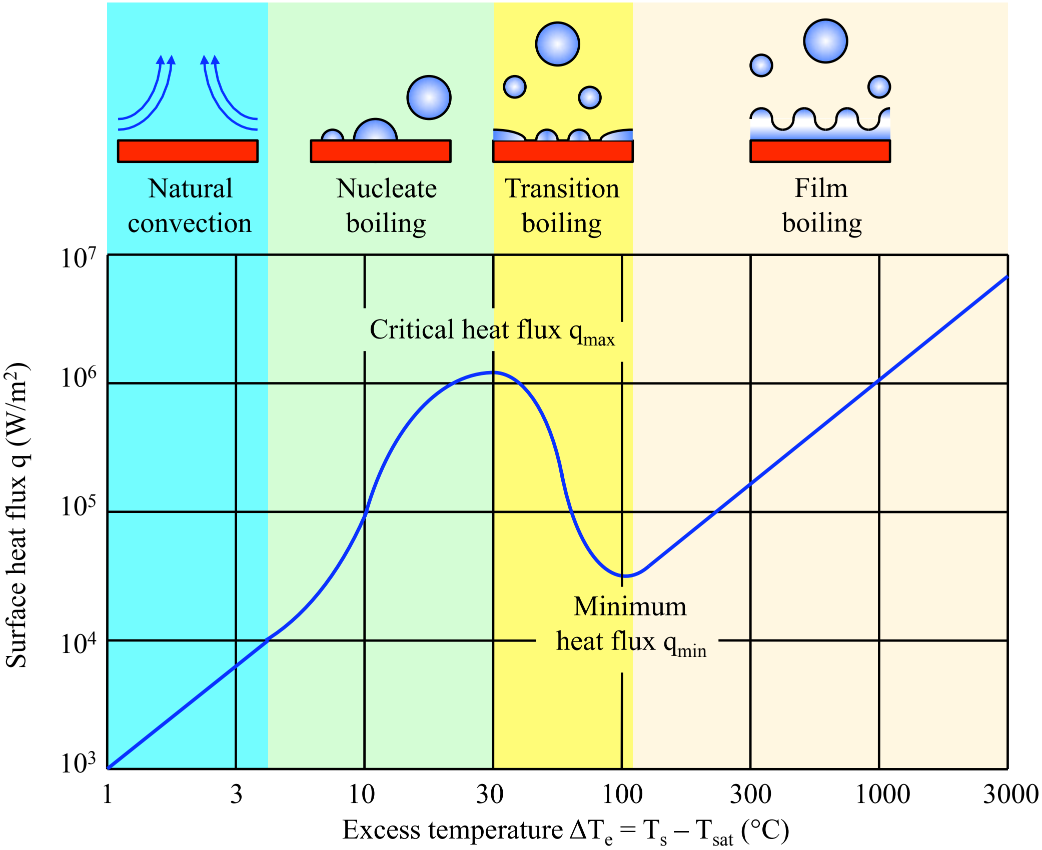


Figure 11. A typical boiling curve for a heated solid surface in contact with water at one atmosphere of pressure. The heat flux q_s is plotted on a log scale versus the excess temperature $\Delta T_e \equiv T_s - T_{\text{sat}}$, where T_s is the surface temperature and T_{sat} is the saturation temperature of the water (100 ^{o}C). The figure shows the behavior of the heat flux in various regimes: free (natural) convection, nucleate boiling, transition boiling, and film boiling.

5.1 Nucleate Boiling

Nucleate boiling occurs for $5^o\text{C} < \Delta T_e < 30^o\text{C}$ for water. (Only free convection occurs for smaller excess temperatures.) Nucleate boiling may be modeled as follows.

Consider a hemispherical gas bubble of radius R forming on top of a horizontal heated surface, as shown in Fig. 12. If σ is the surface tension for the fluid ($\sigma \approx 0.06$ N/m^2 for water), the force holding the hemispherical bubble to the heated surface will be

$$F_{\text{surface tension}} = 2\pi R\sigma \; . \tag{124}$$

This force increases linearly with R as the size of the bubble grows.

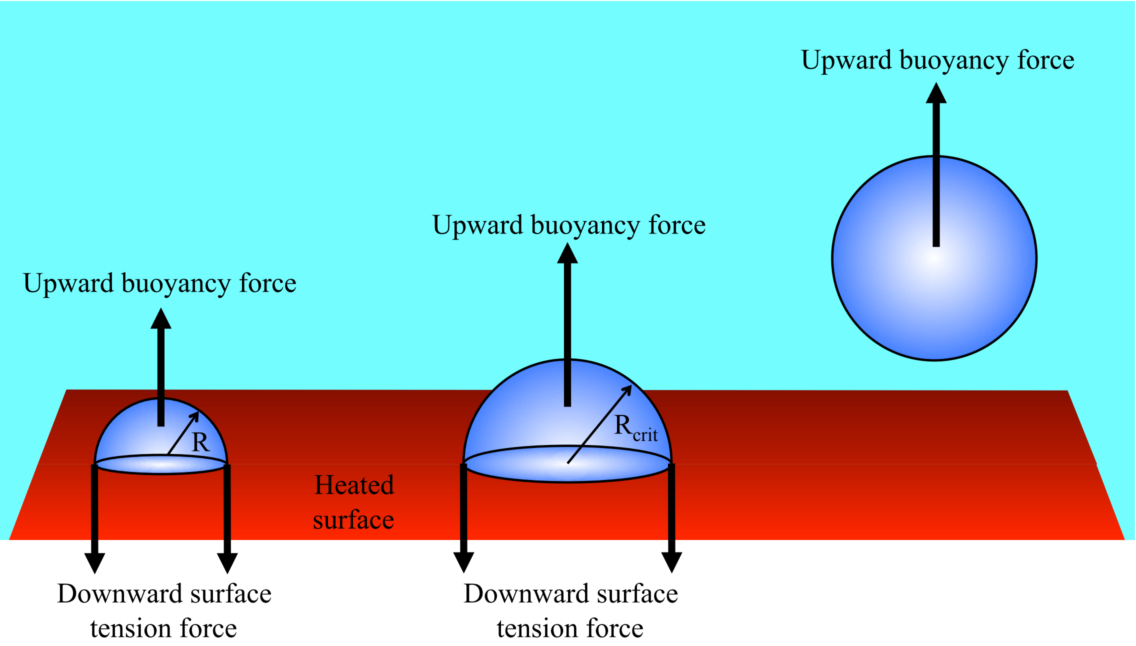


Figure 12. An illustration of nucleate boiling, showing the formation of vapor bubbles on a heated surface immersed in a liquid. The upward buoyancy force on a growing hemispherical vapor bubble of radius $R < R_{\text{crit}}$ is too small to pull the bubble away from the surface; the downward surface tension force on the bubble is strong enough to keep the bubble bound to the heated surface. When the bubble has grown to a size $R = R_{\text{crit}}$, the upward buoyancy force becomes equal to the downward surface tension force, and the bubble is ready to break away from the surface.

On the other hand, the buoyancy force tending to pull the bubble away from the heated surface increases like R^3:

$$F_{\text{buoyancy}} = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) g(\rho_l - \rho_v) \; , \tag{125}$$

in which $g \approx 9.807$ m/sec^2 is the gravitational acceleration, ρ_l is the density of the liquid phase of the fluid, and ρ_v is the density of the gaseous vapor phase of the fluid. For boiling water, $\rho_l \approx 1000$ kg/m^3 and $\rho_v \approx 0.5$ kg/m^3

Since the buoyancy forces increase much faster than the surface tension force as the bubble grows, eventually the bubble will become large enough that its buoyancy pulls it free of the surface. This occurs for a critical bubble radius $R = R_{\text{crit}}$ at which $|F_{\text{buoyancy}}| = |F_{\text{surface tension}}|$:

$$R_{\text{crit}} = \sqrt{\frac{3\sigma}{g(\rho_l - \rho_v)}} \,. \tag{126}$$

Due to conservation of volume, the hemispherical bubble of radius R_{crit} will form a spherical bubble of radius $2^{-1/3}R_{\text{crit}} \approx 0.79R_{\text{crit}}$ when it breaks away from the heated surface, but this change is small enough in comparison with other uncertainties in this crude calculation that it may be ignored.

If drag effects are neglected, once the bubble breaks away from the surface it will accelerate uniformly. Designating M to be the mass of a bubble of uniform density ρ_v, the acceleration may be written as

$$a = \frac{F_{\text{buoyancy}}}{M} = \frac{g(\rho_l - \rho_v)}{\rho_v} \,. \tag{127}$$

It will be assumed that the bubble must rise by at least a distance of $2R_{\text{crit}}$ above the heated surface before a new bubble can form and subsequently be released by the same site on the surface. With uniform acceleration a (distance $= at^2/2$ after time t), the clearance time t_{clear} required for the old bubble to get clear of the surface by a distance of $2R_{\text{crit}}$ is

$$t_{\text{clear}} = \sqrt{\frac{2(2R_{\text{crit}})}{a}} = \sqrt{\frac{4R_{\text{crit}}\rho_v}{g(\rho_l - \rho_v)}} \,. \tag{128}$$

The thermal energy density removed from the heated surface by the bubbles is $h_{fg}\rho_v$, where h_{fg} is the latent heat of vaporization. (For conversion of water to steam, $h_{fg} \approx 2300$ kJ/kg.) Assuming that the heated surface is covered by a layer of bubbles with an average thickness of $R_{\text{crit}}/4$ (bubbles grow slowly at first, so smaller bubbles predominate on the surface) and that these bubbles are removed and replaced by new ones on a timescale of t_{clear}, the maximum heat flux which the bubbles can remove from the surface is

$$\begin{aligned} q_{\max} &\sim \frac{\text{(thermal energy density) (average thickness of bubble layer)}}{\text{clearance time per bubble}} \\ &\sim \frac{(h_{fg}\rho_v)\ (R_{\text{crit}}/4)}{t_{\text{clear}}} = \frac{1}{8}h_{fg}\rho_v\sqrt{\frac{g(\rho_l - \rho_v)R_{\text{crit}}}{\rho_v}} \\ &\sim \frac{3^{1/4}}{8}h_{fg}\rho_v\left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4} \,. \end{aligned} \tag{129}$$

The result of this simple derivation is confirmed by more rigorous calculations and by experimental results [2, 3]:

$$q_{\max} \approx 0.15h_{fg}\rho_v\left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4} \,. \tag{130}$$

The actual numerical constant is somewhat variable, depending on various factors like the number of imperfections on the heated surface which can promote bubble nucleation.

Using the values for water, Eq. (130) gives a maximum heat flux of 1.2 MW/m^2. Thus boiling can transfer an enormous amount of heat energy.

An empirical expression for nucleate boiling heat transfer at lower values of $(T_s - T_{\rm sat})$ is [3]:

$$q \ \approx \ \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2} \left[\frac{c_{p,l}(T_s - T_{\rm sat})}{C_{s,f} h_{fg} {\rm Pr}_l^n}\right]^3 \ . \tag{131}$$

Typically $C_{s,f} \sim 0.01$ and $n \sim 1$, but the precise values of these constants depend on the type of fluid, the type of surface material, and the smoothness of the surface.

5.2 Transition Boiling

As the excess temperature increases through the transition boiling regime (30^oC$<\Delta T_e < 120^o$C for water), the swirling mixture of gas and liquid above the heated surface increasingly impedes bubbles trying to leave the surface. Therefore in the transition boiling regime, the heat transfer rate *decreases* with increasing ΔT_e, ultimately reaching a minimum value at $\Delta T_e \approx 120^o$C. This effect may be incorporated into the simple physical model of boiling by assuming that bubbles rising from the surface are subject to a drag force:

$$F_{\rm drag} \ = \ \frac{1}{2} C_D \left(\pi R_{\rm crit}^2\right) \left(\frac{\rho_l + \rho_v}{2}\right) v^2 \ , \tag{132}$$

in which C_D is the coefficient of drag, $\pi R_{\rm crit}^2$ is the cross-sectional area of the bubble, $(\rho_l + \rho_v)/2$ is the average density of the turbulent mixture of liquid and gas through which the bubble is moving, and v is the velocity of the bubble.

By setting this drag force equal in magnitude to the buoyancy force $F_{\rm buoyancy} = (4/3)\pi R_{\rm crit}^3 g(\rho_l - \rho_v)$, the terminal velocity of the bubbles is found to be

$$v_{\rm term} \ = \ \sqrt{\frac{16 g(\rho_l - \rho_v) R_{\rm crit}}{3 C_D (\rho_l + \rho_v)}} \ . \tag{133}$$

If the drag effects are large enough, a bubble will reach this terminal velocity long before it clears the surface by a distance $2R_{\rm crit}$ to allow for new bubble formation. The clearance time is thus

$$t_{\rm clear} \ = \ \frac{2R_{\rm crit}}{v_{\rm term}} \ = \ \sqrt{\frac{3 C_D (\rho_l + \rho_v) R_{\rm crit}}{4 g(\rho_l - \rho_v)}} \ . \tag{134}$$

This clearance time leads to a minimum heat flux,

$$\begin{aligned} q_{\rm min} \ &\sim \ \frac{h_{fg}\rho_v R_{\rm crit}}{2 t_{\rm clear}} \ = \ \frac{2}{\sqrt{3C_D}} h_{fg}\rho_v \sqrt{\frac{g(\rho_l - \rho_v) R_{\rm crit}}{(\rho_l + \rho_v)}} \\ &\sim \ \frac{2}{3^{1/4} C_D^{1/2}} h_{fg}\rho_v \left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4} \ . \end{aligned} \tag{135}$$

A more rigorous analysis [2, 3] shows that the numerical constant is smaller than indicated in Eq. (135) but that otherwise this result is correct:

$$q_{\rm min} \ \approx \ 0.09 h_{fg}\rho_v \left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4} \ . \tag{136}$$

The numerical constant in Eq. (136) is also rather variable and depends on the specific system.

The phenomenon of boiling is actually more complicated than this simple analysis of bubbles indicates. For example, bubbles commonly join together to form jets and columns of gas spurting up from the heated surface. Moreover, a broad film of gas may be transiently established along the surface, then collapse due to the Rayleigh-Taylor instability of the heavy liquid above the gas film. Fortunately, these effects do not change the functional dependence of the quantities that have been calculated; they only play a role in establishing the numerical coefficients which have been quoted.

To summarize the physical arguments, in transition boiling, there are so many bubbles rising that they slow each other down by a factor of $\sim \sqrt{\rho_v/\rho_l}$ and lead to a minimum heat transfer $q_{\rm min}$ that is smaller than the maximum heat flux $q_{\rm max}$ by this same factor. Using values for water, the minimum heat flux is $q_{\rm min} \approx 20$ kW/m^2, far lower than the maximum $q_{\rm max} \approx 1.2$ MW/m^2.

Transition boiling is a dangerous regime in which to operate a system. As the temperature of the heated surface increases, the heat flux from the surface to the fluid decreases for the reasons which have already been explained. This decrease in heat removal from the surface can cause the surface temperature to increase further. Thus transition boiling systems are unstable to temperature excursions and may accidentally reach the melting temperature of the surface.

5.3 Film Boiling

Film boiling, which occurs for 120^oC$< \Delta T_e$ for water, is almost precisely the mirror image of laminar film condensation. A permanent film of vapor separates the bulk of the fluid from the solid surface, and heat must be transferred across this gaseous layer. Liquid fluid may join the film layer by changing phase, a process involving h_{fg} (just as with condensation). Because of the very strong parallels between film boiling and laminar film condensation, the entire derivation of Section 4.1 may be applied to film boiling provided that the liquid and vapor quantities are interchanged ($\rho_l \leftrightarrow \rho_v$, etc.) and the numerical coefficients are adjusted to match experimental data. Thus the average heat transfer coefficient for film boiling is

$$\overline{h_{\text{film boil}}} \;=\; C \left[\frac{\rho_v(\rho_l - \rho_v) g h'_{fg} \kappa_v^3}{(T_s - T_{sat})\mu_v D} \right]^{1/4} \quad , \tag{137}$$

in which $h'_{fg} \equiv h_{fg} + 0.8 c_{p,v}(T_s - T_{sat})$, L is the characteristic length of the surface, and C is a constant of order unity that depends on the geometry of the surface. For a spherical surface, $L = D$ and $C = 0.67$; for a horizontal cylinder, $L = D$ and $C = 0.62$.

For $T_s > 300$ ^{o}C, radiative heat transfer across the vapor layer is also significant. The heat transfer rates due to boiling and radiation cannot simply be added together, since there is "cross-talk" between the mechanisms: both film boiling and radiation can alter the width of the vapor layer, and both are affected by the width of the vapor layer. The total heat transfer coefficient $\overline{h}$ is [3]:

$$\overline{h}^{4/3} \;=\; \overline{h_{\text{film boil}}}^{\,4/3} + \overline{h_{\text{rad}}}\;\overline{h}^{\,1/3} \quad , \text{ or}$$

$$\overline{h} \;\approx\; \overline{h_{\text{film boil}}} + \frac{3}{4}\,\overline{h_{\text{rad}}} \qquad\qquad \text{for } \overline{h_{\text{film boil}}} > \overline{h_{\text{rad}}} \tag{138}$$

in which $\overline{h_{\text{rad}}}$ is the radiation heat transfer coefficient as derived in the next section.

6 Thermal Radiation

In addition to the means of heat transfer which have already been discussed, heat energy may also be carried by thermal electromagnetic radiation.

6.1 Fundamentals of Thermal Radiation

Types of electromagnetic radiation, from longer to shorter wavelengths, include radio and microwaves, infrared, visible light, ultraviolet, and X- and gamma rays, as discussed in *Electromagnetism* ?.?. With equal validity, the radiation may be regarded either as waves or as particles called photons. The wavelength λ, frequency f, and speed $c \approx 3.00 \times 10^8$ m/sec of the waves or photons are related by

$$\lambda = \frac{c}{f} \tag{139}$$

Hot objects "glow" by emitting a range of thermal electromagnetic radiation. The frequency is proportional to the energy of the waves

$$E = hf \,, \tag{140}$$

in which $h \approx 6.626 \times 10^{-34}$ J·sec is Planck's constant.

In turn, the energy is proportional to the temperature (in degrees Kelvin) above absolute zero:

$$\text{Energy per photon} \sim k_B T \,, \tag{141}$$

in which $k_B \approx 1.380 \cdot 10^{-23}$ J/^{o}K is Boltzmann's constant.

Since frequency is proportional to energy and energy is proportional to temperature, frequency is proportional to temperature, $f \propto T$. Using Eq. (139), this means that the wavelength of thermal radiation is inversely proportional to the temperature:

$$\lambda \propto \frac{1}{T} \tag{142}$$

In other words, the number of waves (or equivalently the number of photons) per length is proportional to the temperature. At higher temperatures, more waves or photons of thermal radiation fit within a given length, as shown in Fig. 13(a).

Therefore, the number of photons that will fit within a three-dimensional volume varies like

$$\text{Number of photons per volume} \propto T^3 \,, \tag{143}$$

as shown in Fig. 13(b).

Using Eqs. (141) and (143), the radiative energy per volume is

$$\begin{aligned} \text{Radiative energy per volume} &= \left(\frac{\text{energy}}{\text{photon}}\right) \times \left(\frac{\text{photons}}{\text{volume}}\right) \\ &\propto T \quad \times \quad T^3 \\ &\propto T^4 \end{aligned} \tag{144}$$

Note the extraordinary implications of Eq. (144)–an object that is twice as hot will emit 16 times more power as thermal radiation.

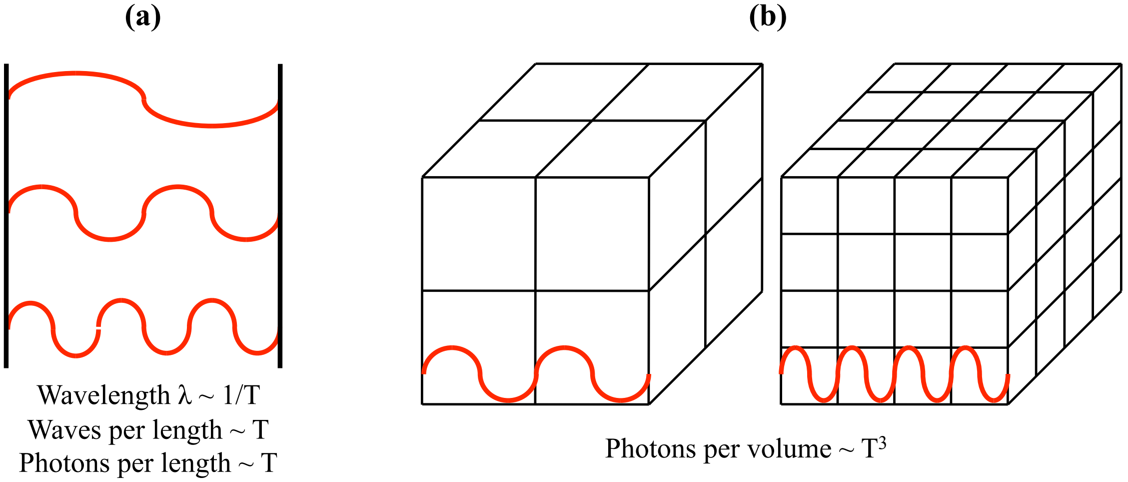


Figure 13. The number of photons of thermal radiation per length or per volume depends on the temperature T above absolute zero. (a). The number of waves or photons per length is proportional to T. **(b)** The number of photons per volume is proportional to T^3.

A more rigorous derivation of thermal radiation (*Statistical Physics* ?.?) confirms Eq. (144) and gives the relevant constant for the heat flux emitted as thermal radiation by a hot object:

$$q = \epsilon \sigma_{SB} T^4 \qquad \textbf{Stefan-Boltzmann law for thermal radiation} \tag{145}$$

in which $\sigma_{SB} \equiv \pi^2 k_B^4 / 60 \hbar^3 c^2 \approx 5.670 \times 10^{-8}$ W m^{-2} (^{o}K)$^{-4}$ is the Stefan-Boltzmann constant and ϵ is the object's **emissivity**. Black surfaces are very good at both emitting and absorbing radiation ($\epsilon = 1$), so thermal radiation is often called "black body radiation." Very reflective surfaces are very poor at both emitting and absorbing radiation ($\epsilon = 0$). Most realistic surfaces are somewhere between these two extremes.

The total emitted power is the product of q from Eq. (145) and the radiating surface's area.

As shown in Fig. 14, the thermal radiation is spread out over the spectrum with a spectral distribution that depends on the temperature. From *Statistical Physics* ?.?, the spectral distribution is

$$\text{Radiant heat flux per wavelength} = \epsilon \frac{2\pi h c^2}{\lambda^5 \exp(hc/\lambda k_B T) - 1} \tag{146}$$

In the most general case, the emissivity is also a function of the radiation's frequency.

The wavelength λ_{peak} at which the greatest amount of radiation is emitted was found in *Statistical Physics* ?.? to be

$$\lambda_{\text{peak}} \approx \frac{2.90 \times 10^{-3}\ \text{m}}{T} \qquad \textbf{Wien displacement law} \tag{147}$$

Therefore the temperature of an object may be estimated by the peak wavelength at which it is glowing (provided that wavelength variations of ϵ are neglected). This result is simply the familiar effect that as an object becomes hotter and hotter, its visible glow is first red, then shifts upward in the spectrum through yellow and finally toward blue. Note that Eq. (147) agrees with the earlier Eq. (142). Table 2 lists the temperature and peak wavelength of several sources of thermal radiation.

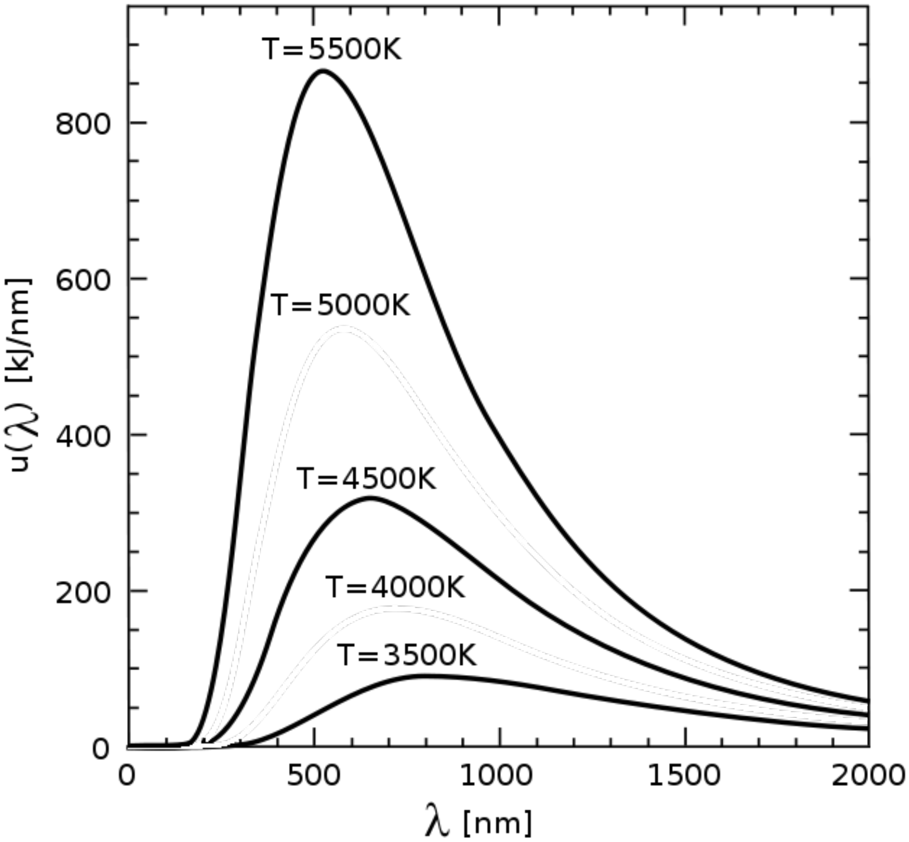


Figure 14. The spectrum of thermal radiation emitted by a hot object depends on the temperature of that object.

Object	**T**	λ_{peak}	**Radiation**
Space	2.7^oK	1.1 mm	Microwave
Human	310^oK	9.4 μm	Infrared
Flame	1800^oK	1.6 μm	Near infrared
Sun	5800^oK	0.5 μm	Visible

Table 2. Thermal radiation from various sources.

The fraction of incident radiation that an object absorbs is the **absorptivity** α. **Kirchoff's law** states that an object's absorptivity is equal to its emissivity, $\alpha = \epsilon$. If an object of temperature T had $\alpha \neq \epsilon$ and were enclosed by radiating surroundings of the same temperature T, the object would have a net gain or loss of thermal energy. Since two things with the same temperature are by definition in thermal equilibrium with each other and hence cannot have a net nonzero transfer of heat energy from one to the other, it is clear that $\alpha = \epsilon$ is required. This same argument applies if the absorptivity and emissivity are wavelength dependent: $\alpha(\lambda) = \epsilon(\lambda)$ for all wavelengths λ.

All incident radiation that is not absorbed must be reflected (assuming no transmission). Thus the reflectivity is $r = 1 - \alpha$.

6.2 Example: Calculation of the Earth's Temperature from Scratch

As an example of the power of this theoretical analysis of thermal radiation, one may estimate the average surface temperature of the earth by using a few basic astronomical values, as illustrated in Fig. 15.

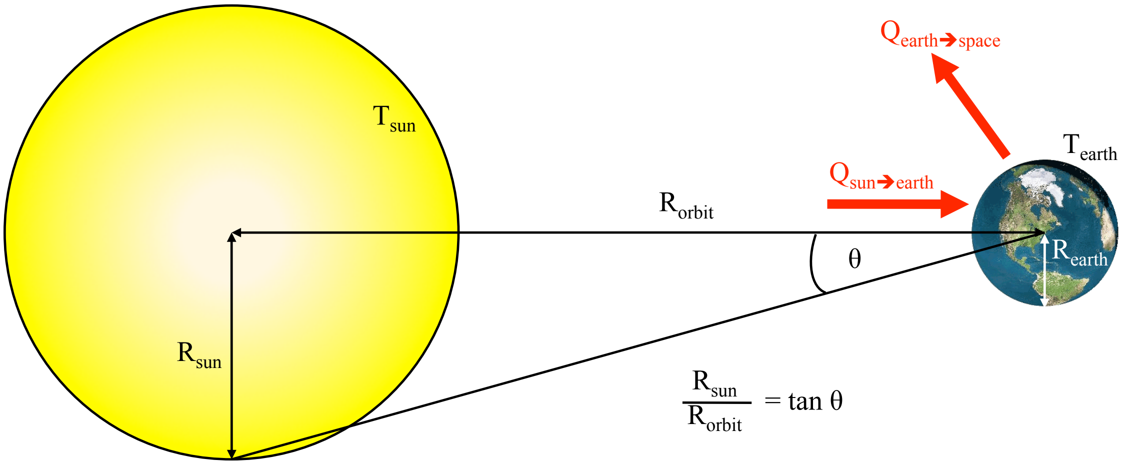


Figure 15. Radiative heat transfer from the sun to the earth. The average surface temperature of the earth may be estimated using a few simple parameters for the sun and the earth.

The first step is to find the sun's surface temperature from the Wien displacement law. The peak solar wavelength occurs near the middle of the visible spectrum, $\lambda_{\text{peak}} \approx 5 \cdot 10^{-7}$ m, so Eq. (147) indicates that the surface temperature of the sun is $T_{\text{sun}} \approx 5800\ ^{o}$K.

If the sun has emissivity $\epsilon_{\text{sun}} \approx 1$ and radius R_{sun}, the total thermal power it radiates will be

$$Q_{\text{sun}} \approx \left(4\pi R_{\text{sun}}^2\right)\left(\sigma_{SB} T_{\text{sun}}^4\right) . \tag{148}$$

It will be assumed that the earth has a radius of R_{earth}, absorptivity and emissivity ϵ_{earth}, and orbits the sun with an orbital radius of R_{orbit}. At the radius of the earth's orbit, the sun's thermal power Q_{sun} is spread out over an area of $4\pi R_{\text{orbit}}^2$. Because the earth's cross-sectional area is πR_{earth}^2, the earth only intercepts and absorbs an amount $Q_{\text{sun}\rightarrow\text{earth}}$ of solar power:

$$Q_{\text{sun}\rightarrow\text{earth}} = \epsilon_{\text{earth}} \frac{\pi R_{\text{earth}}^2}{4\pi R_{\text{orbit}}^2} Q_{\text{sun}} = \epsilon_{\text{earth}} \frac{R_{\text{earth}}^2}{R_{\text{orbit}}^2} \pi R_{\text{sun}}^2 \, \sigma_{SB} T_{\text{sun}}^4 . \tag{149}$$

If the earth's average surface temperature is T_{earth}, the earth will re-radiate to space a power of:

$$Q_{\text{earth}\to\text{space}} \;=\; \left(4\pi R_{\text{earth}}^2\right)\left(\epsilon_{\text{earth}}\sigma_{SB}T_{\text{earth}}^4\right) \;. \tag{150}$$

For the earth to be in thermal equilibrium, the amount of radiant heat it receives must be the same as it emits. Setting $Q_{\text{sun}\to\text{earth}} = Q_{\text{earth}\to\text{space}}$, one finds:

$$T_{\text{earth}} \;=\; \sqrt{\frac{R_{\text{sun}}}{2R_{\text{orbit}}}} T_{\text{sun}} \;. \tag{151}$$

Using $R_{\text{sun}} \approx 7 \cdot 10^8$ m, $R_{\text{orbit}} \approx 1.5 \cdot 10^{11}$ m, and the previously obtained result $T_{\text{sun}} \approx 5800\ ^o$K, the average surface temperature of the earth is found to be

$$T_{\text{earth}} \;\approx\; 280\ ^o\text{K} \;=\; 7\ ^o\text{C}. \tag{152}$$

That's actually a fairly good estimate when you average over the whole planet. It's only about 8^o below the actual value. The actual average temperature is higher than our estimate because our calculation did not include the greenhouse effect that traps solar heat in the earth's atmosphere. For more information on the greenhouse effect and more precise calculations, see *Meteorology* ?.?.

Note that only the ratio $R_{\text{sun}}/R_{\text{orbit}}$ mattered. This ratio could be found from the angular width 2θ of the sun in the sky without resorting to fancy astronomical techniques to measure the two radii separately, since $R_{\text{sun}}/R_{\text{orbit}} = \tan\theta \approx \theta$. The visual half-angle of the sun as seen from the earth is $\theta \approx 4.65 \times 10^{-3}$ rad $= 0.266^o$.

6.3 Fancy Radiative Heat Transfer Calculations

In calculating the radiative heat flow from object i to object j, it is generally necessary to use a view factor F_{ij}, which is defined as the fraction of radiation from surface i that reaches surface j. For the general situation illustrated in Fig. 16, the view factor is:

$$F_{ij} \equiv \frac{1}{A_i} \int_{A_i} \int_{A_j} dA_i dA_j \frac{\cos\beta_i \ \cos\beta_j}{\pi s^2} \ , \tag{153}$$

in which s is the distance the radiation travels between the two surfaces, and β_i and β_j are the angles between the radiation path and lines normal to surfaces i and j, respectively. The reason for including the $\cos\beta_i$ and $\cos\beta_j$ factors is that radiative heat transfer has the greatest effect normal to a surface and has no effect tangent to a surface. The $1/s^2$ factor is included because the radiation intensity falls off like the inverse square of the distance, as shown explicitly in Section 6.2. In order to limit the maximum view factor to 1, the expression must be divided by π, as shown.

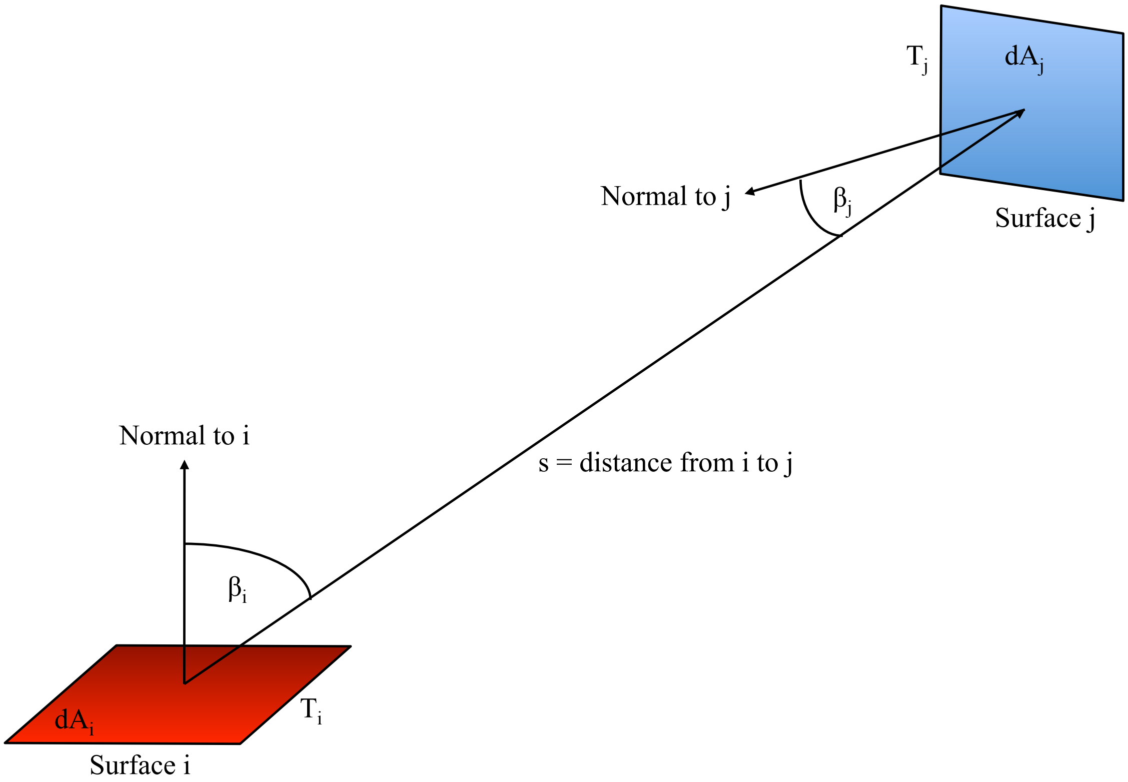
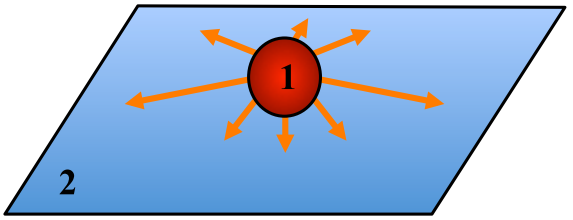


Figure 16. Radiative heat transfer between surfaces i and j. The path taken by the radiation is illustrated and compared with vectors normal to each surface. Radiation which is emitted by one surface but not intercepted by the other surface is not explicitly shown.

View factors for various specific geometries have been calculated and are summarized in Fig. 17 and [2, 3, 5].

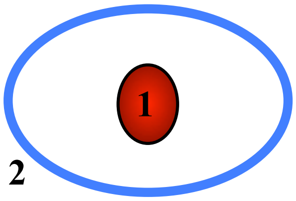
(a) Small surface near large plane



Nearly half of the radiation from surface 1 goes toward plane 2, and the rest goes away from plane 2

$$F_{12} = 1/2$$

(b) Convex surface enclosed within another surface



All radiation from 1 hits 2

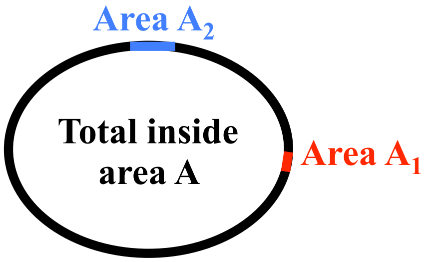
Surface areas A_1 and A_2 govern how much radiation from surface 2 strikes 1

$$F_{11} = 1$$

$$F_{21} = A_1/A_2$$

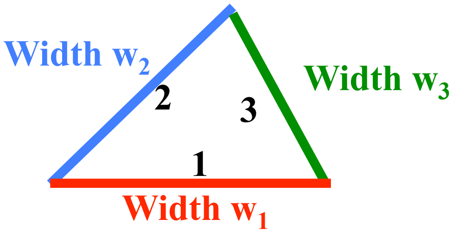
$$F_{22} = 1 - F_{21} = 1 - A_1/A_2$$

(c) Area on inside of concave enclosure



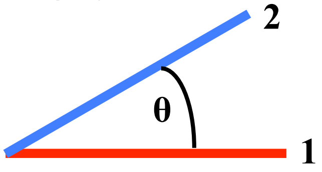
$$F_{12} = A_2/A$$

(d) Three-sided enclosure



$$F_{12} = \frac{w_1 + w_2 - w_3}{2w_1}$$

(e) Long symmetrical wedge



Use half-angle formula

$$F_{12} = F_{21} = 1 - \sin(\theta/2)$$

Figure 17. View factors for radiative heat transfer between surfaces i and j in some simple configurations. See [2, 3, 5] for more examples.

From the definition in Eq. (153) it is clear that the view factor obeys the relations:

$$A_i F_{ij} = A_j F_{ji} \qquad \textbf{reciprocity relation} \tag{154}$$

$$\sum_j F_{ij} = 1 \qquad \textbf{summation rule} \tag{155}$$

The radiative heat flow between black bodies i and j ($\epsilon_i = \epsilon_j = 1$) is:

$$Q_{i \to j} = \sigma_{SB} A_i F_{ij} (T_i^4 - T_j^4) \; . \tag{156}$$

Note that the factor A_i in Eq. (156) cancels the factor $1/A_i$ contained within the definition of F_{ij}. For objects other than black bodies, the emissivities must also be taken into account. As a simple example, consider two infinite plane surfaces 1 and 2 which are facing each other as shown in Fig. 18(a).

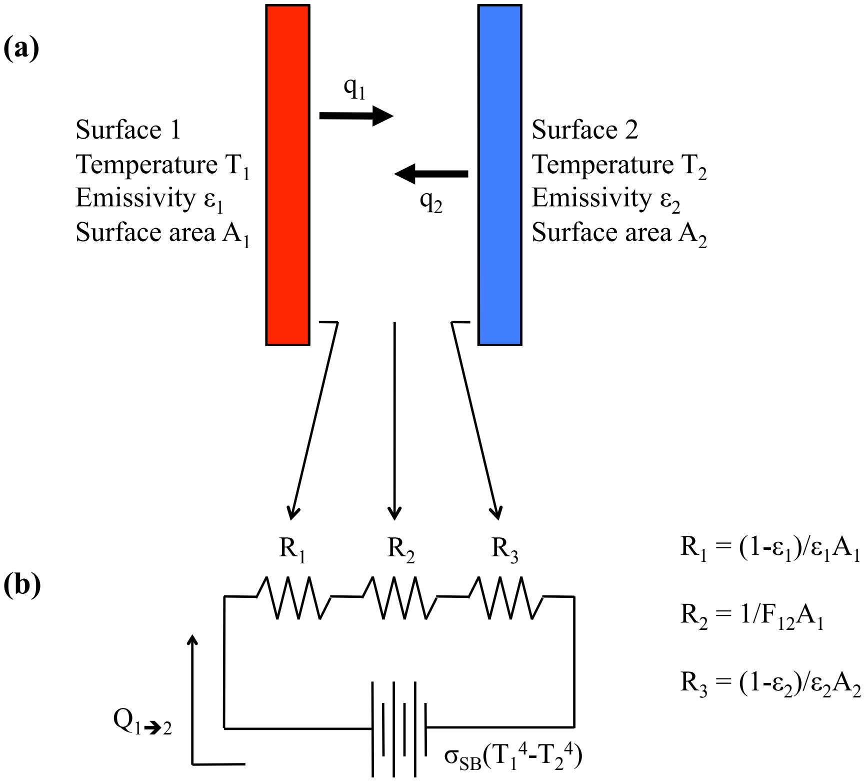


Figure 18. Radiative heat transfer between two parallel plane surfaces. (a) The view factor between the plates is F_{12}. (For infinite plates, $F_{12} = 1$.) **(b)** The radiative heat transfer is analogous to Ohm's law for a circuit, as shown.

Let q_1 and q_2 be the radiant heat fluxes coming from surfaces 1 and 2 respectively in Fig.18(a). q_1 and q_2 include the fresh energy radiated by each surface plus the incoming energy each surface reflects:

$$q_1 = \epsilon_1 \sigma_{SB} T_1^4 + (1-\epsilon_1) q_2 \tag{157}$$

$$q_2 = \epsilon_2 \sigma_{SB} T_2^4 + (1-\epsilon_2) q_1 \; . \tag{158}$$

Combining Eqs. (157) and (158), the net heat flux between the surfaces is found to be:

$$q_{1\to 2} \equiv q_1 - q_2 = \frac{\sigma_{SB}\left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} = \epsilon_1 \epsilon_2 \sigma_{SB} \left(T_1^4 - T_2^4\right) \left[1 + \sum_{n=1}^{\infty} (1-\epsilon_1)^n (1-\epsilon_2)^n\right] \; . \tag{159}$$

Equation (159) shows that $q_{1\to 2}$ may be interpreted as the heat flux $\epsilon_1 \epsilon_2 \sigma_{SB}(T_1^4 - T_2^4)$ which passes directly between the surfaces, plus the reflections of that heat flux which must make one or more round trips bouncing back and forth between the surfaces before they are absorbed.

Alternatively, the Ohm's-law-like interpretation shown in Fig. 18(b) states that the radiation "potential difference" $\sigma_{SB}(T_1^4 - T_2^4)$ must drive the heat flow $Q_{1\to 2} \equiv A_1 q_{1\to 2}$ through three resistances in series: the resistance $(1-\epsilon_1)/\epsilon_1 A_1$ associated with getting through surface 1, the resistance $1/A_1 F_{12}$ between the surfaces, and the resistance $(1-\epsilon_2)/\epsilon_2 A_2$ to get through surface 2:

$$Q_{1\to 2} = \frac{\sigma_{SB}\left(T_1^4 - T_2^4\right)}{\sum \text{resistances}} = \frac{\sigma_{SB}\left(T_1^4 - T_2^4\right)}{\frac{1-\epsilon_1}{\epsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{\epsilon_2 A_2}} \; . \tag{160}$$

For $F_{12} = 1$ and $A_1 = A_2$, Eq. (160) reduces to Eq. (159). Equation (160) (with $F_{12} = 1$) also applies if one surface is completely enclosed by the other. The Ohm's law analogy for radiation is especially useful for complex situations involving $F_{12} \neq 1$ or more than two surfaces [2, 3].

Notice from Eq. (159) that an effective heat transfer coefficient $\overline{h_{\text{rad}}}$ for radiation may be defined:

$$\overline{h_{\text{rad}}} \equiv \frac{q_{1\to 2}}{T_1 - T_2} = \frac{\sigma_{SB}\left(T_1^2 + T_2^2\right)\left(T_1 + T_2\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} \; . \tag{161}$$

References

[1] A. F. Mills, *Heat Transfer* (2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1998). This is probably the best heat transfer textbook. It gives physically intuitive explanations for many heat transfer formulas, as opposed to opaque mathematical derivations or purely empirical formulas with no theoretical justification.

[2] J.H. Lienhard, *A Heat Transfer Textbook* (3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 2003), http://web.mit.edu/lienhard/www/ahtt.html. This is a good introductory textbook. Its step-by-step annotated derivations are very helpful.

[3] Frank P. Incropera, David P. DeWitt, Theodore L. Bergman, and Adrienne S. Lavine *Fundamentals of Heat and Mass Transfer* (6th ed., Wiley, New York, 2006). This is another good introductory heat transfer textbook. It complements Lienhard's book very nicely; things which are omitted or glossed over by Lienhard's book are often treated more thoroughly by the Incropera and DeWitt book, and vice versa.

[4] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (2nd ed., Pergamon Press, New York, 1987), Chapter V. This work covers fluid mechanics in general, and heat transfer in particular, as seen by the brightest Russian theoretical physicists. Depending on your perspective, the result is either the epitome of mathematical elegance or an unsurpassed mathematical nightmare.

[5] E.A. Avallone and T. Baumeister III, *Marks' Standard Handbook for Mechanical Engineers* (9th ed., McGraw Hill, New York, 1987), Sections 4.3 and 4.4. For those who feel compelled to do heat transfer calculations in BTUs and other horrible English units instead of metric units, this book shows how. It has the added virtue that if you get bored with heat transfer, you can wander off and read about everything else from manufacturing techniques to patents.

[6] Wilbur Hankey, *Re-Entry Aerodynamics* (AIAA, Washington, DC, 1988). The clearest, more complete, and most up-to-date book on re-entry heat transfer, which isn't saying much. You also need to translate the contents from units such as BTUs per slug per fortnight into real units.